Variable Quantities and Variable Structures in Topoi

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In memory of my eldest son, William Nevin.

I have organized this chapter into three sections as follows:

1. The conceptual basis for topoi in mathematical experience with variable sets.
2. A formal theory of variable abstract sets as a relativized foundation for geometry and analysis, with due attention to "the" case of constant sets.
3. Sheaves of continuous maps, étendues, and a proposed distinction between variable quantities in particular and variable structures in general.

Readers of Section 1 who are not too familiar with recent work on topoi may find clarification of some concepts in Section 2. Section 3 treats two aspects of sheaf theory not yet sufficiently incorporated into general topoi theory, with some remarks on the possible relevance of their relation to analysis and philosophy.

1.

Around 1963 (the same year in which I completed my doctoral dissertation under Professor Eilenberg's direction) five distinct developments in geometry and logic became known, the subsequent unification of which has, I believe, forced upon us the serious consideration of a new concept of set. These were the following:

"Non-Standard Analysis" (A. Robinson)
"Independence Proofs in Set Theory" (P. J. Cohen)
"Semantics for Intuitionistic Predicate Calculus" (S. Kripke)
"Elementary Axioms for the Category of Abstract Sets" (F. W. Lawvere)
"The General Theory of Topoi" (J. Giraud)

Apart from these specific developments, there has long been in geometry and differential equations the idea that the category of families of spaces smoothly parametrized by a given space X is similar in many respects to the category of spaces itself, and indeed, from the point of view of physics, it is perhaps to such a category with X "generic" or unspecified that our stably correct calculations refer, since there are always small variations or further parameters that we have not explicitly taken into account; the "new" concept of set is in reality just the logical extension of this idea. Of the five specific developments referred to, the decisive one for the concept of variable set was the theory of topoi; while nonstandard analysis, the forcing method in set theory, and Kripke semantics all involved, as will be explained below, sets varying along a poset $\mathbf{X}$, it was Grothendieck, Giraud, Verdier, Deligne, M. Artin, and Hakim who, by developing topos theory, made the qualitative leap—well-grounded in the developments in complex analysis, algebraic geometry, sheaf theory, and group cohomology during the 1950s—to consideration of sets varying along a small category $\mathbf{X}$ and at the same time emphasized that the fundamental object of study is the whole category of sets so varying. Those insisting on formal definitions may thus, in what follows, consider that "variable set" simply means an object in some (elementary) topos (just as, using an effective axiom system to terminologically invert history, we sometimes say that "vector" means an element of some vector space).

Traditionally, set theory has emphasized the constancy of sets, and both Robinson's nonstandard analysis and Cohen's forcing method involve passing from a system $\mathcal{S}$ of supposedly constant sets to a new system $\mathcal{S}'$ that still satisfies the basic axioms for constant sets; however, it is striking that both methods pass "incidentally" through systems of variable sets, and further, that the distinction between the two methods lies in the distinction between two fundamental ways of analyzing variation.

Let us recall what these two ways of analyzing variation are, first in the case of variable quantity. Both involve separating a *domain* of variation and a *type* of quantity (discrete, continuous, scalar, vector, tensor, operator, functional, etc.); let us fix the case R of continuous scalar quantity since the basic distinction between the two analyses is in how the domain of variation is treated. According to the first analysis, the domain X consists of "points" (points of space, instants of time, particles of a body, etc.), and a variable quantity is identified with a mapping $X \to R$; conditions such as continuity or measurability of the variation have to be imposed as additional properties involving additional structure on X. Although the foregoing is the usual view, it is not always adhered to in practice; for example, if X, μ is a measure space, then any member f of the usual $L_p(X, \mu)$ is clearly a variable quantity with domain of variation X, although it makes no sense to speak of the value of f at a point x. The second analysis is to consider that the domain $\mathbf{X}$ consists of parts (subregions of space, subintervals of time, parts of a body, etc.) and that a variable quantity is identified with a lattice homomorphism from parts of R into parts of $\mathbf{X}$; this remains sensible even if parts of $\mathbf{X}$ with μ-null difference are regarded as indistinguishable. The first analysis may be considered as a special case of the second by considering $\mathbf{X} = 2^X$. Conversely, if $\mathbf{X}$ is a complete Heyting algebra we can define its points as the infinitary sup-preserving lattice homomorphisms $\mathbf{X} \to 2$, which if $\mathbf{X} = 2^X$, or more generally if $\mathbf{X}$ is a sober topology for X, will correspond exactly to the mappings $1 \to X$, i.e., to points in the usual sense; of course if $\mathbf{X}$ is measurable sets modulo null sets there often will not be any points. We may also consider as *ideal points* the *finitary* lattice homomorphisms $\mathbf{X} \to 2$, which in case $\mathbf{X} = 2^X$ are just the ultrafilters on X; i.e., the ideal points are the points of the compactification, and the axiom of choice tries to reassure us that at least ideal points exist for any $\mathbf{X}$.

Returning now to nonstandard analysis and forcing, we start with a model $\mathscr{S}$ of a theory of constant sets. If X is a given (say countable) constant set, then $\mathscr{S}^X$ (i.e., all functions from X to $\mathscr{S}$) is a system of variable sets (conforming to the first analysis of variation). If we "stop" the variation at a point of X, we of course get back $\mathscr{S}$; but if we stop ("localize") the variation at an *ideal point* in $\beta(X) - X$, we get a new system $\mathscr{S}'$ of sets that satisfies the same elementary axioms (e.g., those expressing constancy) as $\mathscr{S}$, but which will definitely be different from $\mathscr{S}$. In particular, it will contain new "infinitesimal" elements—the residual traces of the variation that has been "stopped"—shown by Robinson to permit a reduction in the complexity of many definitions and proofs in analysis. For the forcing method we need, however, the second description of variation, applied to variable sets rather than variable quantities; instead of a set X we need a poset $\mathbf{P}$ in $\mathscr{S}$. As was later clarified by Scott and Solovay, it is more invariant though sometimes less convenient to enlarge $\mathbf{P}$ to the Boolean algebra $\mathbf{X}$ of $\neg\neg$-stable elements of the Heyting algebra of all order-preserving maps $\mathbf{P} \to 2$; this $\mathbf{X}$ typically has no points, but we can still consider sets E, which vary along it as follows: for each $A \in \mathbf{X}$, $E(A) \in \mathscr{S}$, and whenever $A \subseteq B$ in $\mathbf{X}$, there is a restriction mapping $E(B) \to E(A)$, and these fit together functorially whenever $A \subseteq B \subseteq C$; moreover, there is the condition

$$E(A) \simeq \prod_{i \in I} E(A_i)$$

whenever $A = \sum_{i \in I} A_i$ is a *disjoint* union in $\mathbf{X}$. If E_1, E_2 are as just described, a "mapping" $E_1 \xrightarrow{\varphi} E_2$ means any family $E_1(A) \xrightarrow[\varphi A]{} E_2(A)$ of mappings in $\mathscr{S}$ indexed by the A in $\mathbf{X}$ and satisfying the commutativity

$$\begin{array}{ccc} E_1(B) & \xrightarrow{\varphi B} & E_2(B) \\ \downarrow & & \downarrow \\ E_1(A) & \xrightarrow{\varphi A} & E_2(A) \end{array} \qquad \text{whenever } A \subseteq B \text{ in } \mathbf{X}$$

Thus we have defined a category $\mathscr{E}$ of sets varying along $\mathbf{X}$ (usually called Boolean-valued sets). Below we will see that it is not necessary to define the traditional ε-relation in $\mathscr{E}$: in any case, the interesting set-theoretic questions such as choice, replacement, the continuum hypothesis, measurable cardinals, etc. are categorical invariants anyway, that is, the questions depend only on how maps compose, not on an a priori notion of iterated membership. We can again localize at any chosen ideal point of $\mathbf{X}$ to obtain $\mathscr{S}'$, a new system of sets that look constant insofar as the most elementary properties [such as axiom of choice, two valuedness (see below)] that distinguish constant from variable sets are concerned, but unlike the previous case of nonstandard analysis, some of the deeper properties that had been proposed to enforce constancy, such as the axiom of constructibility (by taking as $\mathbf{P}$ the basic open sets of the Cantor space) or the continuum hypothesis (by taking $\mathbf{P}$ to be the basic open sets of a big generalized Cantor space), are as Cohen showed destroyed by the passage $\mathscr{S} \rightsquigarrow \mathscr{S}'$ even though "elementary" in the technical sense. It was these examples $\mathscr{S}'$ that led Tierney and me to further generalize the previous theory of topoi in 1969 by making it elementary, since although $\mathscr{S}$, $\mathscr{S}^X$, $\mathscr{E}$ are topoi in the Grothendieck–Giraud sense, the $\mathscr{S}'$ of nonstandard analysis and the $\mathscr{S}'$ of forcing are not; on the other hand, the essential issues of nonstandard analysis and of forcing can be dealt with in a perhaps more natural and certainly more invariant fashion in $\mathscr{S}^X$, respectively $\mathscr{E}$, provided one does not insist on constancy (which here just means on two-valuedness since the axiom of choice and hence the law of excluded middle are already valid in $\mathscr{S}^X$ and $\mathscr{E}$ if they are in $\mathscr{S}$).

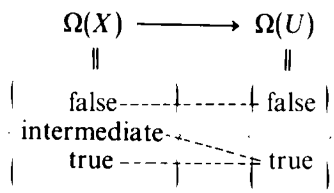
Variable sets arose only "incidentally" in nonstandard analysis and forcing. The original goal was to construct new models of the theory of constant sets bearing nontrivial relation to a given such model. By contrast, in Kripke semantics for the Heyting predicate calculus the variation is essential also in the end result. Indeed, the thrust of Kripke's completeness theorem is that no logic stronger than intuitionistic logic can be valid for sets that are varying in any serious way, and in the other direction the Heyting predicate calculus *is* valid in *all topoi*, although topoi are qualitatively more general in at least two ways than the models for that calculus considered by Kripke in 1963. The latter also involved a system of variable sets $\mathscr{S}^{\mathbf{P}^{op}}$ identified with sets varying along a poset $\mathbf{P}$ according to the second analysis of variation, but more simply than with forcing and Boolean-valued sets as described above. We only consider $E(A) \in \mathscr{S}$ for $A \in \mathbf{P}$ itself and the transition mapping $E(B) \to E(A)$ in $\mathscr{S}$ whenever $A \leq B$ in $\mathbf{P}$; the transitions are subject to functoriality (transitivity), but to no further conditions, and the mappings $E_1 \to E_2$ in $\mathscr{S}^{\mathbf{P}^{op}}$ are defined as before. The interpretation of these variable sets was in terms of subjective variation of knowledge; the elements of $\mathbf{P}$ are called stages of knowledge and $A \leq B$ is taken to mean that A is a deeper (or later) stage of knowledge than B; for any set E we have, at any given stage B, constructed certain elements of E and proved certain equalities between pairs of elements constructed, giving an abstract set $E(B)$; if $A \leq B$ is a deeper stage of knowledge, the transition map $E(B) \to E(A)$ reflects that no constructed elements are ever lost and no proven equations are ever disproved, but the map is neither surjective nor injective since new elements may be constructed and new equalities proved at stage A. Considering that an n-ary relation S on E means simply another variable set equipped with a monomorphic mapping $S \rightarrowtail E^n$ in $\mathscr{S}^{\mathbf{P}^{op}}$, where E^n is the cartesian power in $\mathscr{S}^{\mathbf{P}^{op}}$, one finds that the operation of substitution is easily defined and that the operations of conjunction, disjunction, implication, and universal and existential quantification on relations, which are uniquely defined by the rules of inference, exist in $\mathscr{S}^{\mathbf{P}^{op}}$. The crucial fact is that universal quantification and implication do not commute with evaluation at a stage, but rather

$$\begin{gathered}\forall x\, [S_1(x, y) \Rightarrow S_2(x, y)] \text{ holds at } B, \text{ where } y \in E(B)\\ \text{iff for all } A \text{ with } A \leq B \text{ and for all } x \in E(A)\\ \text{if } S_1(x, y|A) \text{ then } S_2(x, y|A)\end{gathered}$$

so that such a relation's truth value at B depends on its classical truth value at all deeper stages. Since by definition $\neg S = [S \Rightarrow \text{false}]$, a similar situation holds for logical negation, so that in particular $S \Rightarrow \neg\neg S$ but not conversely. Though an infinite $\mathbf{P}$ is required to simultaneously refute all intuitionistically nonprovable inferences of predicate calculus, the two-element poset

$$\mathbf{P} = \{U \leq X\} = \mathbf{2}$$

suffices to refute the inference from $\neg\neg S$ to S; moreover there is an immediate connection with geometry since for $\mathbf{P} = \mathbf{2}$ it is easily verified that $\mathcal{S}^{\mathbf{P}^{\mathrm{op}}}$ is equivalent to the category of $\mathcal{S}$-valued sheaves on a two-point topological space with three open sets. As in any topos there is a unique "set" Ω of truth values in $\mathcal{S}^{\mathbf{P}^{\mathrm{op}}}$, which in the case $\mathbf{P} = \mathbf{2}$ is just the two-stage variable set



the intermediate value reflecting the fact that for an inclusion $S \rightarrowtail E$ in $\mathcal{S}^{\mathbf{2}^{\mathrm{op}}}$, there may be elements of $E(X)$ that are not in $S(X)$ but that do get mapped into $S(U)$ upon "restriction" to $E(U)$. Then $(\neg\neg S)(X)$ consists of all the elements of $E(X)$ that on restriction are in $S(U)$, so that $S \subseteq \neg\neg S$ is in general a proper inclusion. I would like to emphasize that recognizing the central importance for mathematics of the Heyting predicate calculus (i.e., intuitionistic logic) in no way depends on accepting a subjective idealist philosophy such as constructivism; objectively variable sets occur (at least implicitly) every day in geometry and physics and the fact that this variation is reflected in our minds in no way means that it is "freely created" by our minds; but it seems to have been the intuitionists who first succeeded in formulating the logic that holds for at least a certain definite portion of variation in general.

Some idea of which portion may be conveyed by the following example. Suppose the variation is along the temporal ordering and consider the statement:

The feudal landlords are the ruling class.

Recalling that truth is preserved by the transition maps in the sense that once something is true, it remains true, we see that the above statement is not an acceptable relation S in such a hypothetical temporal topos since it was once false, became true for a period, then became false again. On the other hand, consider the statement:

The feudal landlords have ruled.

Although this statement was sometimes true and sometimes false it has, solely by virtue of its grammatical structure, the requisite property that as soon as it became true, it remained forever true. Of course there are also more profound ways of dealing with temporal variation than simply identifying time with a poset that is governing the variation in the simple way we have suggested here, and some of these can even be accounted for by a suitable topos.

In Kripke's topoi existential quantification and disjunction *do* commute with evaluation at stages:

$\exists x\, S(x, y)$ is true at B for $y \in E(B)$ iff there is x in $E(B)$ with $S(x, y)$.

$S_1(x) \vee S_2(x)$ is true at B iff $S_1(x)$ is true at B or $S_2(x)$ is true at B.

To put it more set-theoretically, images of mappings and unions of subsets commute with evaluation at B. But these facts apparently cannot be maintained in modeling Heyting-type theory (i.e., intuitionistic analysis) even though the Kripke topoi do have an intrinsic higher-order structure. At any rate, they certainly are false in the kind of topos that was already well understood in the 1950s, namely in a category of set-valued sheaves on a topological space. Such is also governed by a poset, namely the complete Heyting algebra of open sets of a space, but all the variable sets E satisfy the familiar "pasting" or sheaf condition, which is similar to but more involved than the infinite product condition for *disjoint* coverings mentioned above in connection with Boolean-valued sets. In particular, the image of a map between variable sets is also required to be a sheaf, which has the following effect. Suppose $E_1 \xrightarrow{f} E_2$ is a map of sheaves and $y \in E_2(U)$ for some open set U in the space over which the sets are varying. Then the rules of inference force

> $\exists x[f(x) = y]$ is true on U iff there exists an open covering U_i of U and there exist $x_i \in E_i(U_i)$ such that $f(x_i) = y|U_i$, $i \in I$,

where $y|V \in E(V)$ denotes the restriction of y to $V \subseteq U$. A similar statement holds for disjunction (union of two subsheaves) and these facts (with "covering" suitably interpreted) hold in any topos.

Both the Kripke topoi and the topological topoi are generated by their subsets of 1 in the sense that

> if $S \rightarrowtail E$ is any monomorphism, then either it is an isomorphism or else there exists an object U for which $U \to 1$ is a monomorphism and there exists a morphism $U \xrightarrow{x} E$ that does not factor through S (i.e., $x \notin S$).

Examples of topoi of variable sets for which this condition does not hold were known implicitly for a long time and more explicitly in the 1950s in connection with group cohomology, in the form of the category $\mathcal{S}^G$ of G-sets (permutation representations) for a group G. Here the only subsets U of 1 are 0 and 1, yet a map $1 \xrightarrow{x} E$ is only an element of E that *is fixed by* G; or looking at it from the other side, the most natural object that is the source of enough elements x (as U in the above condition) is G itself acting on itself by translation, yet (if $G \neq 1$) it is not a subset of 1. Now upon taking abelian-group-object categories, we have for a space X

$$\text{Ab}(\text{Sheaves}(X, \mathcal{S})) = \text{abelian sheaves on } X$$

and for a group G

$$\text{Ab}(\mathcal{S}^G) = G\text{-modules}.$$

Moreover, the functor represented by 1 [i.e., $\mathcal{X}(1, \_\_)$] becomes, for $\mathcal{X} = \text{Sheaves}(X, \mathcal{S})$, the global sections functor

$$\text{Sheaves}(X, \mathcal{S}) \xrightarrow{\Gamma} \mathcal{S}$$

and, for $\mathcal{X} = \mathcal{S}^G$, the fixed point functor

$$\mathcal{S}^G \to \mathcal{S},$$

which upon taking Ab and taking right-derived functors via injective resolutions as in Cartan–Eilenberg, become respectively

$$H^n(\mathcal{X}, E) = H^n(X, E), \qquad \text{and} \qquad H^n(\mathcal{X}, E) = H^n(G, E),$$

i.e., cohomology of a space with values in a sheaf and cohomology of a group with values in a G-module become cases of cohomology of a topos with values in an abelian object of the topos.

[An important factor contributing to the nontriviality of cohomology is the nontrivial contradiction between $\exists$ (image) and evaluation at X as mentioned previously. For example, in a Kripke topos in which there is a shallowest level of knowledge, $H^n = 0$ for $n > 0$. However, there are other factors, since without the shallowest level the $H^n(\mathcal{S}^{\mathbf{P}^{\text{op}}}, E)$ occur in algebraic topology and in partial differential equations under the name of "higher inverse limits."]

Note that if X and Y are any sober topological spaces (for example any Hausdorff spaces) then the topos morphisms (to be defined presently)

$$\text{Sheaves}(X, \mathcal{S}) \to \text{Sheaves}(Y, \mathcal{S})$$

are equivalent to the continuous maps $X \to Y$, and that if G and H are any groups, then the topos morphisms

$$\mathcal{S}^G \to \mathcal{S}^H$$

are equivalent to the group homomorphisms $G \to H$. Recalling that group cohomology arose in the geometric context where the group is related to the Poincaré fundamental group of a space, one sees the possible virtues of one big category Top in which spaces and groups are on "equal" footing and in which moreover a space, its universal covering space, and its fundamental group might be connected by morphisms, etc. Such a conception occurred to many people but apparently was still not sufficient to give rise to the general concept of topos; it turns out that there is a large class of topoi, the so-called étendue discussed below, that includes both these classes of examples (spaces and groups) as well as much more, and yet the "typical" topos is of still quite another kind that also arose in algebraic geometry but turned out to have still other connections with logic, apparently in particular with A. Robinson's notion of "forcing in model theory."

Some of these more typical examples arise immediately in algebraic geometry as follows. Let K be a field in $\mathcal{S}$ and let $\mathbf{A}$ be the small category of all finitely presented (finitely generated) commutative algebras over K. The category $\mathcal{S}^{\mathbf{A}}$ of all covariant functors from $\mathbf{A}$ to $\mathcal{S}$ is a topos with the following interesting property: the underlying set functor $R\colon \mathbf{A} \to \mathcal{S}$ is a commutative-K-algebra-object in $\mathcal{S}^{\mathbf{A}}$; if $\mathcal{X}$ is any topos defined over $\mathcal{S}$ and A is any commutative-K-algebra-object in $\mathcal{X}$, then there is a unique continuous map (geometrical morphism of topoi)

$$\mathcal{X} \xrightarrow{f} \mathcal{S}^{\mathbf{A}} \qquad \text{such that} \qquad f^*(R) \xrightarrow{\approx} A.$$

A continuous map of topoi is just a functor having a left exact adjoint f^*. Since tensor products in $\mathbf{A}$ distribute over finite cartesian products, the subcategory $\mathcal{G}$ of $\mathcal{S}^{\mathbf{A}}$ consisting of the functors that preserve finite cartesian products is also a topos, and $R \in \mathcal{G}$; the continuous maps

$$\mathcal{X} \to \mathcal{G}$$

of topoi are equivalent to the commutative-K-algebra-objects A which satisfy moreover the condition

$$a^2 = a \qquad \text{entails} \qquad a = 0 \vee a = 1$$

where the disjunction is interpreted as union of certain subobjects of A in $\mathcal{X}$. The topos $\mathcal{G}$ may also be described as the largest subtopos of $\mathcal{S}^{\mathbf{A}}$ for which the left adjoint to the inclusion functor renders isomorphic the particular inclusion map

$$\{0, 1\} \hookrightarrow A(K \times K, \_\_)$$

in $\mathcal{S}^{\mathbf{A}}$. In turn $\mathcal{G}$ has a largest subtopos $\mathcal{Z}$ for which the inclusion

$$U \cup U_{1-()} \hookrightarrow R$$

is rendered isomorphic, where for a $B \in \mathbf{A}$,

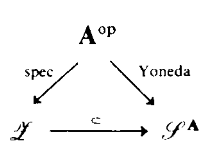
$$(U \cup U_{1-()})(B) = \{b \in B \mid \exists(1/b) \text{ or } \exists[1/(1-b)]\}.$$

The topos $\mathcal{Z}$, sometimes called the "big Zariski" topos, classifies, for any topos $\mathcal{X}$ over $\mathcal{S}$, the commutative-K-algebra-objects A in $\mathcal{X}$ that are *local* rings in $\mathcal{X}$ in the sense that $a_1 + a_2$ unit in A entails

$$(a_1 \text{ unit in } A) \vee (a_2 \text{ unit in } A)$$

where again disjunction refers to union of subobjects (of $A \times A$) in $\mathcal{X}$.

These examples represent a surprising twist of a logic that is not yet fully clarified: while a topos such as Sheaves(X, $\mathcal{S}$) for a topological space X may be "identified" as a *particular* space, a topos such as $\mathcal{G}$ or $\mathcal{Z}$ should be identified rather as a *general* concept of space in the sense that its *objects* correspond to particular spaces as follows: The Yoneda embedding, which factors through $\mathcal{Z}$, may be called "spec," since its image is equivalent to the category of affine algebraic schemes over K



but in fact the whole category of algebraic spaces over K is fully included in $\mathcal{Z}$. For $X \in \mathcal{G}$, the morphism set $\mathcal{G}(X, R)$ is the ring of functions on the space X, and the definition of $\mathcal{G}$ may be considered as the readjustment of the notion of coproduct (from that of $\mathcal{S}^{\mathbf{A}}$) so that $\text{spec}(B_1 \times B_2) = \text{spec}(B_1) + \text{spec}(B_2)$. The latter condition is geometrically reasonable since

$$\mathcal{G}(X_1 + X_2, R) \simeq \mathcal{G}(X_1, R) \times \mathcal{G}(X_2, R)$$

holds in any case for the rings of functions, and if X_1, X_2 are determined by their global function rings B_1, B_2, it is plausible that the "disjoint" union $X_1 + X_2$ is entirely determined by its global function ring as well. Similarly, the definition of $\mathscr{Z}$ is a further readjustment of unions so that finite coverings of $X = \operatorname{spec} B$ by Zariski open subobjects (not only the clopen coverings determined by idempotents) have the correct effect on function rings. Here all Zariski open subobjects are derived from pullback from the basic one

$$\begin{array}{ccc} U & \hookrightarrow & R \\ \| & & \| \\ \mathbf{A}(K[t, t^{-1}], \_) & \hookrightarrow & A(K[t], \_) \end{array}$$

The above examples are typical in that *every* Grothendieck–Giraud topos $\mathscr{Y}$ over $\mathscr{S}$ is the classifying topos for an essentially unique theory in the way that $\mathscr{S}^{\mathbf{A}}$ is the classifying topos for the theory of commutative algebras, $\mathscr{G}$ the classifying topos for the theory of idempotentless algebras, and $\mathscr{Z}$ the classifying topos for the theory of local algebras. That is, for any topos $\mathscr{X}$ over $\mathscr{S}$, the category $\mathrm{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y})$ is equivalent to the category of models in $\mathscr{X}$ of the theory associated with $\mathscr{Y}$. The kind of theories that occur, which we may briefly call "positive" theories, are in general many-sorted ones, which in addition to equational axioms (such as the distributive law) may also involve axioms of the form

$$\varphi_1 \quad \text{entails} \quad \varphi_2,$$

where the φ_1 and φ_2 are formulas built up from "atomic" operations and relations using finite conjunction, possibly infinitary disjunction, and existential quantification. If only finitary disjunctions (which includes the empty disjunction "false") are involved, the topos will be a "coherent" one, and conversely. So far as models in $\mathscr{X} = \mathscr{S}$ are concerned (or more generally for any "Boolean" $\mathscr{X}$ in which classical logic reigns) any first-order theory can be construed as positive by introducing further atomic relations to play the role of any negative formulas that occur in the axioms. In this sense Deligne's theorem that every coherent topos has points is equivalent to the Gödel–Henkin completeness theorem for first-order logic and Barr's theorem that every Grothendieck–Giraud topos has sufficient Boolean-valued points implies Mansfield's Boolean-valued completeness theorem for infinitary first-order theories. However, as already pointed out, classical logic does not apply in even the simplest topological topoi $\mathscr{X}$, so that if conditions φ that essentially involve $\forall$, $\Rightarrow$ are considered, Kripke's method has to be taken into account.

As a simple example of the above, let G be a group in $\mathscr{S}$ and $\mathscr{Y} = \mathscr{S}^G$

the category of all G-sets. It is known that for any topos $\mathscr{X}$ over $\mathscr{S}$

$$\mathrm{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{S}^G) = H^1(\mathscr{X}, G)$$

up to equivalence, i.e., that continuous maps $\mathscr{X} \to \mathscr{S}^G$ correspond to principal homogeneous G-objects in $\mathscr{X}$. The latter are in fact models of the positive theory generated by one unary operation for each element g of G subject to the axioms

$(g_1 g_2)x = g_1(g_2 x)$

$g_1 x = g_2 x$ entails "false", for $g_1 \neq g_2$ (an infinite number of axioms with one free variable)

$\bigvee_{g \in G} [gx = y]$ (a formula with two free variables)

$\exists x[x = x]$

The last axiom means that for a principal homogeneous G-object X in $\mathscr{X}$, the morphism $X \to 1$ is an epimorphism in $\mathscr{X}$; it does not necessarily mean that X has a globally defined element $1 \to X$ in $\mathscr{X}$, which would imply $X \cong p^*(G_1)$ where $\mathscr{X} \xrightarrow{p} \mathscr{S}$ is the canonical grounding or "global-sections" functor and G_1 denotes G acting on itself by left translation.

Another example of central importance is the classifying topos for the theory of equality, i.e., the models of this theory in $\mathscr{X}$ are just the objects of $\mathscr{X}$,

$$\mathrm{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y}) \cong \mathscr{X} \qquad \text{for all } \mathscr{X}.$$

Here

$$\mathscr{Y} = \mathscr{S}^{\mathbf{S}_{\mathrm{fin}}}$$

is the category of all functors from the category of finite sets into the category of sets, with the inclusion functor U as the "generic set adjoined to $\mathscr{S}$." [If we instead consider $\mathscr{S}^{\mathbf{S}_{\mathrm{fin}}^{\mathrm{op}}}$ we get the classifying topos for the theory of Boolean algebras, with $n \rightsquigarrow 2^n$ as the "generic Boolean algebra."] Taking $\mathscr{X} = \mathscr{S}$, we see that the category of points of $\mathscr{S}^{\mathbf{S}_{\mathrm{fin}}}$ is just $\mathscr{S}$, so that in a definite sense the *objects* of any Grothendieck-Giraud topos $\mathscr{X}$ may be identified with *continuous* mappings

$$\mathscr{X} \to \mathscr{S}^{\mathbf{S}_{\mathrm{fin}}}$$

from the "space" $\mathscr{X}$ into the "space of all sets." There is a clear analogy here, to which we will return in a moment, between variable sets and variable quantities: If we replace $\mathscr{S}^{\mathbf{S}_{\mathrm{fin}}}$ by the space of real or complex numbers, we would obtain the well-known correspondence between algebras of variable quantities and spaces; however, that correspondence is perfect only for compact spaces, while for "algebras" (topoi) of variable sets the correspondence is perfect for all sober spaces and even, as we have seen, for vastly more general "spaces" as domains of variation for the variable sets. Such a "space" may be considered as the space of models for a theory, in a more refined sense than the usual one since a point determines a model, not only an elementary-equivalence class of models. It is useful to consider that the analysis of the domain of variation is a categorical refinement of the analysis in terms of parts, in the sense that any object of the topos may also be considered a generalized "part" of the domain of variation. Under the analogy the functors f^*, which are just those preserving small direct limits (addition) and finite inverse limits (multiplication), correspond to algebra homomorphisms.

We can even speak of *ideal points* of a topos in the following way. Any given infinitary theory can be construed as a finitary theory in which all formulas obtained by infinitary disjunction are reconsidered to be "atomic" formulas; in topological terms we relax the sheaf condition to consider only finite coverings. This leads to a sort of Wallman compactification

$$\mathscr{X} \subseteq \overline{\mathscr{X}}$$

for any G–G topos $\mathscr{X}$ (with $\overline{\mathscr{X}}$ a coherent topos), and we may consider any point $\mathscr{S} \xrightarrow{x} \overline{\mathscr{X}}$ of $\overline{\mathscr{X}}$ to be an ideal point of $\mathscr{X}$. The axiom of choice for $\mathscr{S}$ (i.e., the Deligne–Gödel–Henkin theorem) reassures us that enough ideal points exist for any $\mathscr{X}$. The above inclusion has the special properties that it generates $\overline{\mathscr{X}}$ in the sense that any $\overline{X}$ in $\overline{\mathscr{X}}$ is the canonical direct limit of all the objects X in $\mathscr{X}$ that map in $\overline{\mathscr{X}}$ to $\overline{X}$, and that the inclusion preserves *finite direct* limits (i.e., its derived functors vanish). Thus for any ideal point p of $\mathscr{X}$, the composite $\mathscr{X} \hookrightarrow \overline{\mathscr{X}} \xrightarrow{p^*} \mathscr{S}$ is a functor that preserves both finite inverse limits and finite direct limits; factoring this functor by a method due to Kock and Mikkelsen should lead to a "localization" functor $\mathscr{X} \to \mathscr{S}'$, which preserves even higher-order logic, with $\mathscr{S}'$ a two-valued topos but, in general, *not* a Grothendieck-Giraud topos (i.e., not defined over $\mathscr{S}$), making precise the idea that constancy is a limiting case of variation, but constancy is not an entirely determinate concept.

We need not always use the syntactical machinery of logic in presenting the classifying topoi $\mathscr{Y}$, since as has been elegantly utilized by Joyal and Wraith, every Grothendieck–Giraud topos over $\mathscr{Y}_0 = \mathscr{S}$ can be constructed by a finite number of applications of the following three operations:

(1) Adjoining a generic element of a given object Y_0 of a given topos

$\mathscr{Y}: \mathscr{Y}' = \mathscr{Y}/Y_0 =$ (the category whose morphisms are the commutative triangles in $\mathscr{Y}$ ending in Y_0) has the property that for any given continuous map $\mathscr{X} \xrightarrow{p} \mathscr{Y}$,

$$\mathrm{Top}_{\mathscr{Y}}(\mathscr{X}, \mathscr{Y}') \cong \mathscr{X}(1, p^* Y_0).$$

Here, if we denote by $\Pi: \mathscr{Y}' \to \mathscr{Y}$ the continuous map $\prod_{Y_0}$ whose inverse image part is just $\Pi^* = (\ ) \times Y_0$, the "generic element of Y_0" is just the element in $\mathscr{Y}'$

$$\begin{array}{ccc} 1_{\mathscr{Y}'} & \to & \Pi^*(Y_0) \\ \| & & \| \\ Y_0 & \to & Y_0 \times Y_0 \end{array}$$

determined by the diagonal map.

(2) Adjoining a generic family of objects indexed by a given object I of a given topos $\mathscr{Y}$. That is, there is a topos $\mathscr{Y}''$ with a continuous map $\mathscr{Y}'' \xrightarrow{q} \mathscr{Y}$ such that for any topos $\mathscr{X}$ and given continuous map $\mathscr{X} \xrightarrow{p} \mathscr{Y}$,

$$\mathrm{Top}_{\mathscr{Y}}(\mathscr{X}, \mathscr{Y}'') \cong \mathscr{X}/p^*(I)$$

and of course for $J \in \mathscr{X}$ we consider $\mathscr{X}/J$ as the topos of families of objects of $\mathscr{X}$ smoothly parametrized by J. In case $I = 1$, $\mathscr{Y} = \mathscr{S}$ we have $\mathscr{Y}'' = \mathscr{S}^{\mathbf{S}_{\mathrm{fin}}}$ as discussed earlier.

(3) Inverting a given morphism $Y_1 \xrightarrow{y} Y_2$ in a given topos $\mathscr{Y}$. There is a subtopos $\mathscr{Y}''' \hookrightarrow \mathscr{Y}$ (constructed with help of a Grothendieck modal operator in $\mathscr{Y}$) such that for any continuous $\mathscr{X} \xrightarrow{p} \mathscr{Y}$,

$$p \text{ factors through } \mathscr{Y}''' \quad \text{iff} \quad p^*(y) \text{ is an isomorphism in } \mathscr{X}.$$

These constructions imply some others:

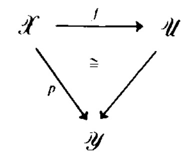
(4) Adjoining a generic morphism between two given objects Y_1, Y_2 in a given topos $\mathscr{Y}$. There is $\mathscr{Y}^{\mathrm{IV}} \to \mathscr{Y}$ such that for any continuous $\mathscr{X} \xrightarrow{p} \mathscr{Y}$,

$$\mathrm{Top}_{\mathscr{Y}}(\mathscr{X}, \mathscr{Y}^{\mathrm{IV}}) \cong \mathscr{X}(p^* Y_1, p^* Y_2).$$

(5) Adjoining a generic *epi*morphism between two given objects [by applying (3) to the image of a generic morphism].

(6) Imposing equality of two given morphisms $Y_1 \rightrightarrows Y_2$ [by applying (3) to the inclusion morphism of their equalizer], etc.

In the above we have denoted by $\mathrm{Top}_{\mathscr{Y}}(\mathscr{X}, \mathscr{U})$ the category whose objects are pairs $\langle \mathscr{X} \xrightarrow{f} \mathscr{U}, \theta \rangle$, where θ is an isomorphism of functors.



In the constructions (1) and (4) this category of continuous maps is equivalent to the small discrete category determined by the set of morphisms on the right-hand side of the condition, while in case (2) it is equivalent to the "large" (and nondiscrete) topos $\mathscr{X}/p^*I$.

For example, we may consider the functor $N:\mathbf{S}_{\text{fin}} \to \mathscr{S}$ which is constantly a countable set; it is the natural number object for the object classifier $\mathscr{Y}_1 = \mathscr{S}^{\mathbf{S}_{\text{fin}}}$ in the sense that is uniquely described as the object satisfying primitive recursion. Thus by (1) the category $\mathscr{S}^{\mathbf{S}_{\text{fin}}}/N$ has both a generic number n and a generic "set" U. The "set" of natural numbers less than n is a definite object $[n]$ of the last category, so by (5) we can further adjoin a generic epimorphism $[n] \twoheadrightarrow U$ to obtain the classifying topos $\mathscr{Y}_2$ for the theory of *explicitly finite* sets, i.e., for any $\mathscr{X} \xrightarrow{p} \mathscr{S}$, an object of the category

$$\mathscr{F}_2(\mathscr{X}) = \text{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y}_2)$$

"is" a triple $\langle A, k, r\rangle$, where A is an object in $\mathscr{X}$, k a natural number *in the sense of* $\mathscr{X}$, and $r:[k] \twoheadrightarrow A$ an epimorphism in $\mathscr{X}$. It is easily shown that the canonical map $\mathscr{Y}_2 \xrightarrow{q} \mathscr{Y}_1$ has q^* *faithful*, so that the image of q in the sense of $\text{Top}_{\mathscr{S}}$ is $\mathscr{Y}_1$ itself. But what of finite sets in themselves, i.e., those objects for which "there exists" an enumeration by some natural number?

For example, in the two-stage Kripke topos $\mathscr{S}^{\mathbf{2}^{\text{op}}}$ (i.e., the topos of sheaves on the two-point space with three open sets), there is no need for coverings in interpreting "there exists," so a finite object turns out to mean $E(X) \to E(U)$ such that both $E(X)$ and $E(U)$ are finite in the sense of $\mathscr{S}$ *and* the restriction map $E(X) \to E(U)$ itself is *surjective*. Note that a natural number $[k]$ in $\mathscr{S}^{\mathbf{2}^{\text{op}}}$ must have *identity* $[k] \to [k]$ as its restriction, so that the existence of an enumeration for E by some $[k]$ does *not* imply that E has a one-to-one enumeration by a $[k]$ (not even locally).

On the other hand if we consider $A = \{z \mid |z| = 1\}$, the unit circle in the complex plane, as a locally connected Hausdorff topological space, then in $\mathscr{X} = \text{Sheaves}(X, \mathscr{S})$ coverings definitely do play a role; the natural number $[2]$ is the sheaf corresponding to the étale space $X + X \to X$ over X consisting of two disjoint copies of the circle, one above the other, but if we consider the étale space $X \xrightarrow{z^2} X$ (which may be pictured as a single double loop) then the corresponding sheaf E is *locally* enumerated by (even locally isomorphic to) [2] and hence is finite.

As a third example consider $\mathscr{X} = \mathscr{S}^G$, the topos of G-sets. Here a natural number $[k]$ is a finite set with *trivial* G-action, but since there is a "covering" on which $\mathscr{S}^G$ becomes equivalent to $\mathscr{S}$ itself, a finite object is just an arbitrary finite G-set.

Our definition of finite is equivalent to the following definition, independent of the concept of natural number, which was studied by Kock, Lecouturier, and Mikkelsen. E is finite iff it is a member of the smallest subset of the power set of E, which contains 0 and all singletons, and is closed with respect to binary unions. If $\mathscr{X} \xrightarrow{f} \mathscr{Y}$ is any continuous map and Y is finite in $\mathscr{Y}$, then f^*Y is finite in $\mathscr{X}$, a property that is manifest for our definition but in fact also valid without the existence of an object N of all natural numbers in $\mathscr{Y}$. Thus taking the category $\mathscr{F}(\mathscr{X})$ of finite objects is a contravariant category-valued 2-functor of the topos $\mathscr{X}$.

Just as the enlargement

$$\mathbf{A}^{\mathrm{op}} \subseteq \mathscr{X} \subseteq \mathscr{S}^{\mathbf{A}}$$

of the category of affine schemes was necessary because there are algebraic spaces (notably Grassman manifolds) that are not determined by a single global ring of variable quantities, so a hypothetical enlargement

$$\mathrm{Top}_{\mathscr{S}} \subseteq \mathscr{H} \subseteq \mathrm{CAT}^{\mathrm{Top}^{\mathrm{op}}}$$

of the 2-category of Grothendieck–Giraud topoi may be necessary to account for very general domains of variation that are not determined by a single global topos of variable sets. From the above discussion of finiteness, we derive the following hypothetical property of $\mathscr{H}$: A map $\mathscr{F}_2 \xrightarrow{q} \mathscr{F}$ is "epic" in $\mathscr{H}$ if for every $\mathscr{X} \in \mathrm{Top}_{\mathscr{S}}$ and for every $E \in \mathscr{F}(\mathscr{X})$ there exist an $\mathscr{S}$-set S_i of objects of $\mathscr{X}$ with $\Sigma_i S_i \to 1$ epic and $E_i \in \mathscr{F}_2(\mathscr{X}/S_i)$ with

$$q(E_i) \cong \Pi_i^*(E) \qquad \text{in} \quad \mathscr{F}(\mathscr{X}/S_i)$$

It is not clear what the "logic," i.e., the structure of subobject lattices, for a "2-dimensional topos" such as $\mathscr{H}$ should be. Note that the lattice of *subtopoi* of a given topos is actually an *anti*-Heyting algebra; i.e., it is like the system of closed subsets (rather than open subsets) of a topological space. Thus rather than an implication operator right adjoint to conjunction, the lattice of subtopoi has a logical subtraction operator left adjoint to disjunction, so that if we define $\neg\mathscr{A}$ for a subtopos $\mathscr{A}$ to mean $1\backslash\mathscr{A}$ where 1 denotes the whole topos, then it may be useful to consider, by analogy with closed subsets of a space, that

$$\partial \mathscr{A} = \mathscr{A} \wedge \neg \mathscr{A}$$

is the "boundary" through which $\mathscr{A}$ and $\neg\mathscr{A}$ pass over into each other.

The ubiquity of variable sets suggests the following modest but rather definite conceptual guide: Endeavor to do calculations in all branches of mathematics in such a way that insofar as possible they will be valid in an arbitrary topos, not only in "the" category of constant sets, for this in many cases proves to permit a direct application of essentially naive set-theoretic techniques to higher mathematical problems in a new way that was not possible before the work of Grothendieck, Giraud, *et al.*

To pursue further the analogy with quantities, recall that many constructions that are possible for constant quantities, such as the exponential function e^x, remain meaningful and useful in any Banach algebra; similarly many constructions, notably the power set $\mathscr{P}(X) = \Omega^X$, that are possible for constant sets remain meaningful and useful in any topos. [*Caution*: $\mathscr{P}(X)$, unlike e^x, does not commute with evaluation at a point in general.] Partly independently of the fact that any commutative ring can be analyzed as consisting of local-ring-valued functions on its spectrum, it is important for many reasons to develop *linear algebra* (and hence quadratic forms, commutative algebra, Lie algebra, etc.) over an arbitrary commutative base ring. Such a development, of course, has more attendant subtleties (some of which are measured by homology) than the case when the base ring is the field of rational numbers or complex numbers. In a similar way, it is useful to develop *mathematics* over an arbitrary base topos, and this partly independently of the analysis of its objects as constant-set-valued continuous maps on some domain of variation.

Naturally there is not only an analogy but also an inclusion: Just as a system of constant quantities constitutes a (structured) constant set, so a system of variable quantities constitutes a (structured) variable set, usually with the same domain of variation. For example, if the usual construction of the complex numbers from the natural numbers is carried out in the topos of sheaves on a space, what results is just the sheaf of germs of continuous complex-valued functions on the same space. Here the "usual" construction is understood to involve defining the reals as two-sided Dedekind cuts in the rationals; one of the attendant subtleties is that one-sided Dedekind cuts constitute the sheaf of germs of *semicontinuous* real valued functions, which is in fact the natural recipient of the distance function for variable metric spaces and even of the norm for many variable C^*-algebras. Cauchy sequences in the rationals lead only to *locally constant* real-valued functions on the space.

As an example of the kind of "new direct set-theoretic approach" to a problem that is possible, consider a given continuous map $E \to X$ of topological spaces (which may be considered as a family of spaces E_x smoothly parametrized by $x \in X$). If the topos $\mathscr{X} = \text{Sheaves}(X, \mathscr{S})$ is considered as the base "set theory," then as is fairly well understood E is just "one discrete set" *in* $\mathscr{X}$ if the map is étale. But what if the map is arbitrary? In any "set theory" we can, with due care (*geometrically* motivated) to the attendant subtleties of intuitionistic logic, develop general topology, sheaf theory, etc.; then (again modulo some subtleties) the induced continuous map

$$\mathscr{E} \to \mathscr{X}$$

on sheaf categories is equivalent to the global sections functor for the category $\text{Sheaves}_{\mathscr{X}}(E^{\#}, \mathscr{X})$ for a single topological space $E^{\#}$ *in* the set theory $\mathscr{X}$, with $E^{\#}$ in general not (even internally) discrete.

2.

The reader may have wondered why I mentioned my axiomatization of the category of *constant* sets as one of five developments leading to the consolidation of the concept of *variable* set. The reason is that in the development of my own thinking and in that of some of my colleagues, it was necessary to first purify the constant sets of an extraneous mental variation (along ordinals with attendant proliferation of iterated membership chains), which had been reflected in the theory of them and which is of quite a different nature from the more seriously mathematical or physical variation discussed in Section 1, in order to reveal them more starkly for what they are as well as to bring their theory closer to a reflection of actual mathematical practice. (The last was uppermost in my mind when I developed the theory ETCS for an undergraduate course in the foundations of analysis at Reed College.) Moreover, so far only the category language has successfully acted as a basis of unity for people who are trying to get clear on the general workings of variable sets (although elegant theories in the membership language have been given for the special case of topoi that can *be generated by their truth values* [subsets of 1]) and ETCS seems to have been the first categorical set theory.

There can be no doubt that in mathematical practice both sets and their membership as well as mappings and their composition play basic roles. But in setting up a formal theory one should also try to get clear on which of these is primary and which is secondary in mathematical practice.

The traditional view that membership is primary leads to a mysterious absolute distinction between x and $\{x\}$, to agonizing over whether or not the rational numbers are literally contained in the real numbers, to the "discovery" that an *ordered* pair of elements in turn *has* elements which are, however, not the original elements, and to debates over whether the members of the natural number 5 are 0, 1, 3, 4 or not, and all that is clearly just getting started; on its own formal face, a membership-based theory of sets is potentially littered with an infinite number of such formulas that even set theorists refrain from writing down due to their good mathematical sense. This situation, along with a very analogous situation with respect to the standard formalization of predicate logic, has led to the widespread view that a formalized theory and the calculations that it tries to unify are necessarily so sharply divorced from each other that only a pedant would attempt to actually use a formalized set theory, which view only helps to isolate from most people the actual advances set theorists and logicians have made.

Rejecting this pessimistic view, we may try to isolate instead the features of membership-as-primary that lead to nonmathematical set-theoretic "questions" of the kind listed. I believe the conclusion is that membership-as-primary entails membership as *global and absolute* whereas in practice membership is *local and relative*; that is, in practice we only consider membership as a relation between elements of a given set and subsets of the same given set, not between two sets given in vacuo, and the meaning of membership may vary or not vary when we transform the element and the subset according to some mapping that is, for the given context, tautological.

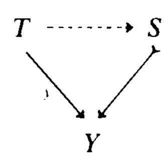
These considerations lead one to formulate the following "purified" concept of (constant) *abstract set* as the one actually used in naive set-theoretic practice of modern mathematics: An abstract set X has elements each of which has no internal structure whatsoever; X has no internal structure except for equality and inequality of pairs of elements, and has no external properties save its cardinality; still an abstract set is more refined (less abstract) than a cardinal number in that it does have elements while a cardinal number does not. The latter feature makes it possible for abstract sets to support the external relations known as *mappings*, which constitute the second fundamental concept of naive set theory (cardinal numbers would admit only the less refined external relations expressed by one being less than another or not). Thus "mapping" is too fundamental to be formally "defined," although we remark that a mapping satisfies the familiar $\forall x \exists ! y$ condition (and prove later that a mapping $X \to Y$ may be represented by its "graph," which is a subobject of a cartesian product $X \times Y$ as well as by its "cograph," which is a quotient object of the (disjoint) sum $X + Y$). The third concept is that of *composition* of mappings, which is defined only in case the codomain of the first mapping is the same abstract set as the domain of the second mapping (indeed otherwise the abstractness of the sets would be violated). Of course composition is *associative*, and there is an *identity* mapping for each set. Thus we have "the" category $\mathcal{S}$ of abstract sets and may speak of isomorphisms, monomorphisms, etc. It has been found possible and effective to express all structure of mathematical interest on a set (or sets) by means of given mappings.

By a "subset" we mean not a set but any monomorphic mapping, i.e., a mapping that does not permit the definition of any structure on its domain. If in the category $\mathcal{S}/Y$ of sets "over" a given set Y we restrict attention to the subsets of Y, we find that there is at most one morphism of $\mathcal{S}/Y$ between any pair of them, defining a reflexive and transitive relation $\subseteq_Y$ among the subsets of Y; thus an inclusion is something simpler and more precise than a condition such as $\forall y[y \in S_1 \Rightarrow y \in S_2]$ since it is rather a single actual mapping that respects the two subsets, although its existence will imply such a condition in view of the following definition of membership.

Suppose $T \xrightarrow{y} Y$ is any mapping and $S \rightarrowtail Y$ a subset. By abuse of notation the letter S names the subset, not only its domain. Then we define

$$y \in S$$

to mean that there exists a mapping (unique by the definition of monomorphism) rendering commutative the following diagram



By axioms for set theory we simply mean certain properties that mathematical experience has shown to be true of sets and mappings and that are judiciously chosen so that together the axioms will imply all other properties that mathematical experience shows to be true. A fundamental axiom is the existence of one-element sets, denoted ambiguously by 1 and characterized by the property that the obvious functor

$$\mathcal{S}_{/1} \xrightarrow{\sim} \mathcal{S}$$

is an isomorphism of categories. We call a mapping $1 \xrightarrow{y} Y$ a global (or eternal) element of Y, but sometimes refer to an arbitrary $T \rightarrow Y$ as "an element of Y defined over T."

The only possible use of abstract sets T is the possibility of indexing or parametrizing things by the elements of T in the hope of clarifying actual relations between the things by means of calculations on mappings introduced to mirror the relations, and the axioms of set theory are based on the need to have abstract sets capable of parametrizing or perfectly parametrizing *mathematical* "things" that have arisen in the course of the mathematical practice of clarifying actual things and their relationships. For example, the elements of a set Y are mathematical "things" and it is precisely the mappings $T \to Y$ that may be used to parametrize these things by the elements of T; in this case perfect parametrization is possible by choosing $T = Y$ and the identity mapping.

A second kind of "thing" that we immediately need to parametrize by T is the (abstract) sets themselves; this can be accomplished by considering mappings $E \to T$ and their fibres E_t, as can be made precise with the help of additional axioms discussed below. While some such parametrizations may be half-perfect in the sense that $E_{t_1} \cong E_{t_2} \Rightarrow t_1 = t_2$, no fixed T however large can parametrize all sets, even up to isomorphism. A question that has been of much "foundational" interest, though of hardly any significance for the practice of algebra, topology, functional analysis, etc., is whether, for a given T, all imaginable families of sets parametrized by T can be represented by $E \to T$ for some E and some mapping; if "imaginable" is interpreted to mean "definable," an affirmative answer to this question is essentially equivalent (for abstract, constant sets) to the postulation of the so-called "replacement schema," whereas if $\mathcal{S}$ is considered as an object in some larger realm, an affirmative answer means that $\mathcal{S}$ itself has "inaccessible cardinality." However, in view of practice and in view of the role of $\mathcal{S}$ as a limiting case of the general notion of continuously variable sets, it seems appropriate to simply define "an internal-to-$\mathcal{S}$ T-parametrized family of objects of $\mathcal{S}$" to mean just a morphism of $\mathcal{S}$ with codomain T.

The problem of perfectly parametrizing ordered pairs of elements is solved by the axiom of cartesian products

$$\frac{T \xrightarrow{\langle y_1, y_2 \rangle} Y_1 \times Y_2}{T \xrightarrow[y_1]{} Y_1,\ T \xrightarrow[y_2]{} Y_2}$$

where projection and diagonal mappings aid in effecting the perfect parametrization and where general elements defined over arbitrary T are needed for a generally effective characterization. Note that in itself $Y_1 \times Y_2$ is just as abstract as any other set; but it has the correct cardinality and the given projection mappings enable it to be considered as having a "rectangular" structure so that its elements can be named in the usual way with help of elements of Y_1, Y_2. An important role of cartesian products is to allow consideration of algebraic operations on the elements of Y as mappings $Y \times Y \to Y$.

The immediate consideration of topoi of more variable sets is helpful even in clarifying constant sets. For example, the functor

$$\mathscr{S} \xrightarrow{T \times (\ )} \mathscr{S}/T$$

enables interpretation of a (relatively) constant set as a particular case of a "set varying over T" (that is, a family of sets parametrized by T) and in particular, identification of an element of Y defined over T in $\mathscr{S}$ with a global element of Y defined over 1_T in $\mathscr{S}/T$.

Given $X \overset{f_1}{\underset{f_2}{\rightrightarrows}} Y$ inducing a structure on X, the elements $T \xrightarrow{x} X$ for which $f_1 x = f_2 x$ can be parametrized by the domain of the *equalizer* E, which is a subset of X satisfying

$$x \in E \quad \text{iff} \quad f_1 x = f_2 x.$$

In terms of 1, $\times$, and equalizers, various constructions such as pullbacks, graphs, intersections of subsets, inverse image of a subset along a mapping, etc. can be carried out and related to each other. In particular, if $E \to T$ is considered as a family of objects parametrized by T, then the "individual" objects in the family can be extracted by means of *pullbacks*

$$\begin{array}{ccc} E_t & \longrightarrow & E \\ \downarrow & & \downarrow \\ T' & \underset{t}{\longrightarrow} & T \end{array}$$

(in particular, consider the case $T' = 1$).

While mappings in themselves are *not* elements of any abstract set (they have, after all, "internal structure"), the mappings between two given abstract sets can be perfectly parametrized by a set of suitable cardinality with the help of an evaluation mapping. Leaving perfection aside for a moment, note that *any* mapping

$$T \times X \xrightarrow{e} Y$$

may be identified with a parametrized family of mappings $X \to Y$, since for any given $1 \xrightarrow{t} T$ we may define e_t by

$$\begin{array}{ccc} X & \xrightarrow{\quad e_t \quad} & Y \\ & {\scriptstyle t_X, 1} \searrow \quad \nearrow {\scriptstyle e} & \\ & T \times X & \end{array}$$

where t_X is the constant composite $X \to 1 \xrightarrow{t} T$. Then the "exponential" set Y^X, which is the right size to parametrize perfectly all the mappings $X \to Y$, is characterized by transformation rule

$$\frac{T \to Y^X}{T \times X \to Y}$$

with the evaluation $Y^X \times X \to Y$ corresponding to the case $T = Y^X$ with the identity mapping above the line. A usual slogan for interpreting the above transformation rule (often given the mysterious name "λ-conversion") says "a function of two variables is equivalent to a function-valued function of one of the variables," although of course the "name" $1 \to Y^X$ of a mapping $X \to Y$ is not "equal" to the latter but is just as abstract as any element of any set in $\mathcal{S}$. An important use of the operation Y^X is to permit consideration as mappings of functionals $Y^X \to Z$ and operators $Y^X \to B^A$; in particular, composition of mappings can itself be studied locally (i.e., for given A, B, C) as a single mapping

$$B^A \times C^B \to C^A.$$

To emphasize the basic nature of the exponentiation functor, let us consider the problem of representing the mechanical motion of matter and calculating: the distance of a particle from a fixed reference point at any given time, the motion of the center of mass of a body, and the velocity of a particle at any time. To this end, suppose M parametrizes particles in a material body (solid or fluid), E parametrizes points of space, T instants of time, and R the positive quantities. Distance may then be represented by a mapping $E \times E \xrightarrow{d} R$, which in particular gives E a convexity structure. The object E^M then parametrizes all possible (and possibly some impossible) placements of the body in space, and one of the consequences (usually calculated with the help of the theory of integration of the mass distribution intrinsic to the body) is a functional

$$E^M \to E$$

called the center of mass. If the motion of the body is represented by a mapping expressing the placement of the body at each time

$$T \to E^M$$

then this is useful (even necessary) to compute the position of the center of mass at any time by composing the two mappings. However, suppose $1 \xrightarrow{p} E$ is a fixed point and we want to calculate the distance from p to any particle at any time; then we must use the exponential transformation rule to express the same motion instead as a mapping

$$T \times M \to E,$$

which we can then compose with $E \xrightarrow{d_r} R$ to find the mapping $T \times M \to R$ to be calculated. But now note that since $T \times M \simeq M \times T$, the same motion can also be expressed as a mapping

$$M \to E^T$$

assigning to each particle the *path* it follows; indeed the motion *must* be so expressed if we are to be able to compose it with the differentiation operator

$$E^T \xrightarrow{(\;)^{\cdot}} V^T,$$

where V is the vector space of translations of E, in order to be able to transform back and thus compute

$$M \times T \to V,$$

the mapping expressing the velocity of any particle in the body at any time. I have discussed this example in some detail to emphasize the elementary character of the exponential adjointness, its necessity for science, the fact that it retains the same form for topoi other than abstract sets in which all mappings are smooth (or even more general categories), and that it must be retained in some fashion even when we consider objects that do not consist mainly of points.

Other crucial facts that follow from the existence of exponentiation, although they do not mention it, are distributive laws like

$$T \times (X_1 + X_2) \xleftarrow{\simeq} T \times X_1 + T \times X_2$$

when "coproducts" (i.e., "disjoint" sums) exist, as they do in topoi. More general laws (involving pullback instead of only the special case $\times$) follow similarly from the fact (additional axiom if you wish) that the pullback functors

$$\mathcal{S}/T' \leftarrow \mathcal{S}/T$$

determined by any $T' \to T$ have *right adjoints* $\prod_t$, which fact is equivalent to the fact that each category $\mathcal{S}/T$ of T-parametrized families of sets has its own internal exponentiation satisfying the same transformation rule over T as the one already written for the case $T = 1$.

The other main use of abstract sets is to parametrize perfectly various *types of quantity*, notably positive real quantities (by R, as mentioned but not fully characterized above), and in particular matter, space, time, and the more problematical truth values (denoted by Ω in a general topos and by 2 in the case of abstract sets). We also customarily use an abstract set N to "perfectly" parametrize the finite discrete quantities. Claiming that the latter can be parametrized by an object N does not make it less problematical, since while we see (bounded pieces of) R every day, no one has yet completed a potential infinity like N. Indeed the introduction of N has given rise to all sorts of unphysical "counter examples" in analysis in the past 100 years. In this latter connection, I may remark that Professor Eilenberg's beautiful lecture on the utter simplicity of the machine needed to compute approximations to a space-filling curve need not convince one that the machine will find enough paper to complete its calculations nor that such a curve exists, since it may rather serve to clarify doubts that N exists (as opposed to 0, 1, 2, ..., each of which does of course exist). One of the reasons for referring to the quantity-type Ω as problematical is that in conjunction with reasonable properties of R it implies the existence of N.

The characterizing property of the set Ω, which perfectly parametrizes the truth values of a topos $\mathcal{X}$, is that there is a distinguished eternal truth value $1 \xrightarrow{\text{truth}} \Omega$, the inverse image $\{X \mid \varphi\}$ of which along any $X \xrightarrow{\varphi} \Omega$ is of course a subset of X for which

$$x \in \{X \mid \varphi\} \quad \text{iff} \quad \varphi x = \text{true}_T, \text{ for any } T \xrightarrow{x} X,$$

but which moreover is such that *any* subset of any X is of this form for a *unique* φ. This implies that Ω is a Heyting-algebra in $\mathcal{X}$, i.e., there is

$$\Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

such that for $\alpha, \varphi, \psi\colon X \to \Omega$

$$\{X \mid \varphi\} \subseteq \{X \mid \alpha \Rightarrow \psi\} \quad \text{iff} \quad \{X \mid \varphi\} \cap \{X \mid \alpha\} \subseteq \{X \mid \psi\},$$

where the intersection and inclusions are as subsets of X. Still simpler is the mapping

$$\Omega \times \Omega \xrightarrow{\wedge} \Omega$$

representing intersection, which is simply the "characteristic function," in the sense of the above axiom, of the subset

$$1 \xrightarrow{\langle \text{true}, \text{true} \rangle} \Omega \times \Omega.$$

The object $P(Y) = \Omega^Y$ perfectly parametrizes the subsets of Y in the sense that the

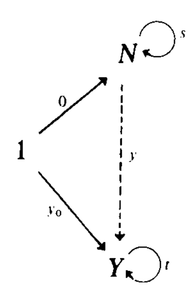
$$T \to \Omega^Y$$

transform biuniquely into relations from T to Y (i.e., into arbitrary subsets of $T \times Y$). Indeed the foregoing sentence together with pullbacks implies the existence of $\prod$, $+$, etc., and so could be taken as the most economical axiom system for topoi (without mention yet the other quantity-types R, N, etc.). In particular, there are for each Y, mappings

$$P(Y) \times Y \xrightarrow{\ni} \Omega$$

that "locally" at Y express fully as a mapping the relation of membership between any subset of Y and any element of Y. Moreover, it follows that existential and universal quantification can themselves be expressed as mappings, e.g., by $Y^X \xrightarrow{\text{image}} \Omega^Y$ and by $PPY \xrightarrow{\cap} PY$.

The "simple recursion" property sufficient to characterize in a topos $\mathscr{S}$ a completed discrete infinity N with successor s and starting point 0 is simply the universal mapping property



i.e., on any set Y any simple transition t and any starting point y_0 determine a unique sequence $N \xrightarrow{y} T$ satisfying two simple recursion conditions. This implies for a topos $\mathscr{S}$ that the forgetful functor

$$\mathrm{Mon}(\mathscr{S}) \to \mathscr{S}$$

from the category of all monoids in $\mathscr{S}$ has a left adjoint W, which is the "word algebra" functor with $W(1) = N$, and a monoid homomorphism "length" $W(T) \to N$ for any set T in $\mathscr{S}$. An even stronger consequence (i.e., not equivalent in the absence of the exponention axiom true in a *topos* $\mathscr{S}$) is that for each T, the forgetful functor

$$\mathscr{S}^{(T)} \to \mathscr{S}$$

has a left adjoint that may in fact be written $W(T) \times Z \leftarrow Z$. Here $\mathscr{S}^{(T)}$ denotes the category whose objects are pairs Y, θ where $T \times Y \xrightarrow{\theta} Y$ is an arbitrary $\mathscr{S}$-mapping; this category will itself be a topos provided $\mathscr{S}$ is a topos *and* N exists in $\mathscr{S}$. The adjointness expresses the expected recursion property for "generalized arithmetic" with a family T of "successors" and a family Z of "zeros." As mentioned in Section 1, a striking consequence of having N in a topos $\mathscr{S}$ is the existence of a topos $\mathscr{S}[U]$ defined over $\mathscr{S}$ such that for any topos $\mathscr{X}$ defined over $\mathscr{S}$ (i.e., equipped with a continuous map $\mathscr{X} \to \mathscr{S}$) there is an equivalence

$$\mathrm{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{S}[U]) \cong \mathscr{X}$$

of categories. It is not known at present whether, conversely, the existence of such an "object-classifying" topos over $\mathscr{S}$ implies the existence of a natural-number parametrizer N in $\mathscr{S}$.

So far we have in fact stated only axioms that in the metatheory need only the extremely weak "logic" of Descartes' analytic geometry, i.e., considering the mappings in $\mathscr{S}$ as the elements of the "universe" of discourse, we have not needed anything so powerful as so-called standard logic with its operators $\Rightarrow$, $\forall$, $\exists$, $\vee$, but rather we have expressed all the axioms only in terms of ordered pairs (triplets, etc.) of mappings subject to certain given functional operators defined on certain equationally defined "varieties" in $\mathscr{S}^2$, $\mathscr{S}^3$, etc. using in principle only substitution; yet we have expressed all the axioms (with the exception of functor "associated champ," which probably should be added) for the general theory of variable sets. This is possible chiefly because all the axioms have the form of "adjointness" except for the most tautological ones, which are purely equational (always understanding that "equational" means identities hold on certain equationally defined "subvarieties"). However, in order to express the two further axioms that in the main express how the notion of constant set contrasts with the general notion of variable set, we need to introduce into the metatheory the logical operators "there exists" and "or" in an *essential* way (I may have used these words before in this section, but always, I believe in ways that can be easily eliminated by standard methods). However, it may prove important to remark that we *still* do not use the operators $\forall$ and $\Rightarrow$ in any essential way, so that the resulting theory [equivalent to the elementary theory of the category of (constant) sets] is still a positive theory in the sense of Section 1, so that the full method of classifying topoi (i.e., generalized "spaces" of models) can in principle still be applied to this metatheory just as for, say, the theory of local rings. The two further axioms are in fact just statements claiming a close unity between the external operators "there exist" and "or" introduced into the set theory (they are written explicitly on the first page of every traditional book of formal set theory) on the one hand, and the internal operators $\exists$ and $\vee$ that exist as *mappings* in any topos $\mathscr{S}$. Explicitly, these axioms are the axiom of choice and two-valuedness:

(AC) For any $X \xrightarrow{f} Y$, if $1_Y \subseteq \exists_f(1_X)$, i.e., if f is epi, then *there exists* g such that $f \circ g = 1_Y$.

(2 val) For any $1 \xrightarrow{\varphi_1} \Omega$, $1 \xrightarrow{\varphi_2} \Omega$, if $\varphi_1 \vee \varphi_2 =$ true, then $\varphi_1 =$ true *or* $\varphi_2 =$ true, where $\vee$ denotes the mapping $\Omega \times \Omega \xrightarrow{\vee} \Omega$ representing union of two subobjects, easily defined equationally with the aid of the infinite intersection mapping $PPY \xrightarrow{\cap} PY$.

Now it is a theorem (from Diaconescu) that the axiom of choice implies the law of the excluded middle, i.e., $1 + 1 \xrightarrow{\sim} \Omega$, i.e., the subobject lattice of any object is Boolean. Thus the second axiom "2 val" (given an absolute interpretation of the external "or") expresses just that there are only two subsets of a one-element set.

It is also a theorem (easy) that the axiom of choice implies that the Boolean algebra of subsets of 1 generates the whole topos. However, although the use of "for any" and "if, then" in the statement of the above axioms is superfluous (i.e., as Gentzen sequents their structure is so simple that the whole system of rules of inference for the metatheory could easily be set up to eliminate these phrases entirely), to state, on the other hand, for a general (not necessarily Boolean) topos what "generating" means (for a class of objects, not necessarily a class of subobjects of 1) requires an *essential* use of $\forall$, $\Rightarrow$ in the metatheory, so that classifying topoi in themselves will reveal less about such a theory. Namely, if $\mathscr{A} \subset \mathscr{E}$ is a given class of objects in a topos $\mathscr{E}$, then "$\mathscr{A}$ generates $\mathscr{E}$" just means that the law of extensionality holds in $\mathscr{E}$ for elements defined on objects A in $\mathscr{A}$, i.e.,

($\mathscr{A}$ gen) For any two subsets S_1, S_2 of any Y, if *for every* A in $\mathscr{A}$ and *every* $A \xrightarrow{y} Y$, $y \in S_1$ *implies* $y \in S_2$, then $S_1 \subseteq_Y S_2$.

3.

In this part all topoi will be over a fixed base $\mathscr{S}$, which we may assume is constant sets to make some things easier to state. Remember, meanwhile, that the basic philosophy is to try to allow $\mathscr{S}$ to be as general as possible in the hope of applying results directly to the case where $\mathscr{S}$ is (the set-valued sheaves on) a topological space, or is (the permutation-representations of) a group, or indeed where $\mathscr{S}$ itself is "a category of spaces" whose points are typical algebras in which "functions" on the "spaces" have their values. Recall that a continuous map $\mathscr{X} \to \mathscr{Y}$ means a functor having a left exact left adjoint and that "$\mathscr{X}$ has a set of generators" means that there is an object in $\mathscr{S}$ that parametrizes a family of objects of $\mathscr{X}$, which in turn satisfies (some internal version of) the extensionality condition stated for an unparametrized class $\mathscr{A}$ at the end of Section 2.

Suppose $Y_{\mathscr{X}}$ is an object of $\mathscr{X}$; what could it mean in terms of the topoi involved that $Y_{\mathscr{X}}$ is actually the sheaf of germs of continuous maps $\mathscr{X} \to \mathscr{Y}$, imagining for the moment that $\mathscr{Y}$ "is" a topological space? A reasonable definition would seem to be the following: for any object U of $\mathscr{X}$, there is a natural equivalence of categories

$$\mathrm{Top}_{\mathscr{S}}(\mathscr{X}/U, \mathscr{Y}) \cong \mathscr{X}(U, Y_{\mathscr{X}}).$$

Here we recall that if $\mathscr{X}$ "is" also a space and if it happens that $U \subseteq 1$, then $\mathscr{X}/U$ "is" just U considered as a space in its own right, while $\mathscr{X}(U, Y_{\mathscr{X}})$ is the set of sections over U of the sheaf $Y_{\mathscr{X}}$. The fact that the left-hand side of the equation above is a category, usually not discrete, makes us realize that we should have taken $Y_{\mathscr{X}} \in \mathrm{Cat}(\mathscr{X})$ as a category object in $\mathscr{X}$ and not just a plain object in general; this is necessary even if $\mathscr{Y}$ is a non-Hausdorff sober space, since relations of the kind $y' \in \overline{\{y\}}$ give rise to (in that case unique) morphisms $y' \to y$ of points. Thus we in general expect $Y_{\mathscr{X}} \in \mathrm{Cat}(\mathscr{X})$.

Now we put the question: Which $\mathscr{Y}$ will have a sheaf of germs of continuous functions $\mathscr{X} \to \mathscr{Y}$ for all $\mathscr{X}$ or for all $\mathscr{X}$ having a set of generators? It is clear that for most $\mathscr{Y}$ this will not be the case since $\mathscr{X}(U, Y_{\mathscr{X}})$ has to be a *set* (parametrizable by an object of $\mathscr{S}$) whereas for $\mathscr{Y}$ = the Zariski topos, or most any classifying topos, there is a proper class of models for the associated theory, and hence $\mathrm{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y})$ is not equivalent to a set, even for $\mathscr{X} = \mathscr{S}$ ("1 point space $\mathscr{X}$"). On the other hand, if $\mathscr{Y}$ is a topological space but $\mathscr{X}$ *arbitrary*, it is easily seen that

$$\mathrm{Top}_{\mathscr{S}}(\mathscr{X}, \mathscr{Y}) \subseteq \mathscr{S}(\mathscr{Y}(1, \Omega_{\mathscr{Y}}), \mathscr{X}(1, \Omega_{\mathscr{X}})),$$

and with slightly more effort one sees that $Y_{\mathscr{X}}$ exists. Indeed since $\mathscr{S}$-valued points of $\mathscr{Y}$ play no particular role, it is irrelevant whether $\mathscr{Y}$ has enough points, so we may assume simply that $\mathscr{Y}$ is generated by its subobjects of 1 [i.e., $\mathscr{Y}$ is the canonical sheaves on the complete Heyting algebra $\mathscr{Y}(1, \Omega_{\mathscr{Y}})$ in $\mathscr{S}$] and get the same result. But there are still many more $\mathscr{Y}$ for which $Y_{\mathscr{X}}$ "always" exist.

Dropping the significant but entirely distinct question of sufficiency of points, we generalize Grothendieck's definition of *étendue* to obtain the definition we will use:

Definition. $\mathscr{Y}$ is an étendue over $\mathscr{S}$ iff there exists $C \twoheadrightarrow 1$ (epic) in $\mathscr{Y}$ such that $\mathscr{Y}/C$ is generated (over $\mathscr{S}$) by its subobjects of 1.

Briefly, the slogan is that $\mathscr{Y}$ is "locally a topological space." In general, to define a property as holding locally one has to allow the covering C to be of the form $C = \sum C_i$ and ask for the property on each $\mathscr{Y}/C_i$; however the property of being a topological space is "additive" so we can use the simpler notion of covering. The first example of an étendue seems to have been the space of moduli of algebraic curves, which is prevented from being globally a space due to the action of Galois groups within each point. Yes, something vaguely reminiscent of particle spin is going on in such spaces, and the most naked form is that for any group G, the category $\mathcal{S}^G$ of G-sets is an étendue with only one point! This is easily seen from the observations that $\mathcal{S}^G/G \cong \mathcal{S}$ and that $G \twoheadrightarrow 1$ where the last two G's denote the regular representation.

A general explanation of why étendues arise in topology is that the "inclusion" functor

$$\text{top}(\mathcal{S}) \xrightarrow{\text{sh}} \text{Top}_{\mathcal{S}}$$

does not in general preserve coequalizers; in particular, suppose a group G acts on a space X and consider the coequalizer diagram for the notion of orbit space—then if the action is good in some recognized sense, $\text{sh}(X//G)$ will also be the (2-) coequalizer in $\text{Top}_{\mathcal{S}}$, while if the action is bad the latter coequalizer tends to be an étendue that is *not* a space. (It seems to me that functional analysis *internally* in such a topos may shed light on hard cases in harmonic analysis and ergodic theory, but I have not had time to investigate this in detail.)

Actually the group case is not quite typical, contrary to what is suggested by some exercises in SGA4; if $\mathcal{Y}$ is any topos having a set $\mathcal{A}$ of generators such that for $A, A' \in \mathcal{A}$, $\mathcal{Y}(A, A')$ consists entirely of *mono*morphisms, then $\mathcal{Y}$ is an étendue, since $C = \sum_{A \in \mathcal{A}} A$ works. For example, consider the topos $\mathcal{S}^{\circlearrowleft}$ whose objects are just sets each equipped with an arbitrary endomorphism, for which $\langle N, s\rangle$ is a convenient generator; then

$$\mathcal{S}^{\circlearrowleft}/\langle N, s\rangle \cong \mathcal{S}^{\omega}$$

where ω denotes the ordered set of natural numbers; the topos $\mathcal{S}^{\omega}$ even has enough points to be a topological space—these points are just the natural numbers plus one more point at infinity whose stalk functor is

$$\mathcal{S} \xleftarrow{\lim} \mathcal{S}^{\omega}.$$

This shows that $\mathcal{S}^{\circlearrowleft}$ is an étendue, which in fact has exactly two points $\{\mathbb{Z}\}$ = (the image under the projection of the point at ∞) and $\{N\}$ = (the image under the projection of each and every finite point n of the "covering" space); not all endomorphisms in the category of points are isomorphisms though.

Now it is easy to make the conjecture that the only $\mathcal{Y}$ that "always" have sheaves $Y_{\mathfrak{x}}$ of germs are the étendue. If so, study of both classes could perhaps be deepened. We have seen that many "domains of variation" are too big to be sets (but it would be even worse to consider them as "abstract classes," for example, since part of their essence is a very strong 2-topological character expressed by the concept of a topos or something like it). On the other hand perhaps, at least the idea that every *type of quantity* is a set could be maintained ("arbitrary cardinals" as a "type of quantity" would actually be quite far from mathematical practice) then the old ideas of Descartes, and more explicitly Riemann, that "every" domain $\mathscr{Y}$ of variation is isomorphic to a part of a type of quantity could be retained simply as the definition of a particular kind ("quantitative" as a special case of "qualitative") of domain; but such a definition in the present context would seem to reduce to our condition that $Y_{\mathscr{X}}$ exist for all $\mathscr{X}$. The idea that, for example, a cohomology class $\mathscr{X} \xrightarrow{\alpha} \mathscr{S}^G$ is a sort of variable quantity of type G varying over $\mathscr{X}$ has a definite intuitive appeal, in spite of the fact that α vanishes at every *point* of $\mathscr{X}$.

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