

AXIOM OF CHOICE AND COMPLEMENTATION

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ABSTRACT. It is shown that an intuitionistic model of set theory with the axiom of choice has to be a classical one.

A topos \mathfrak{E} is a category which has *finite limits* (i.e. finite products, intersections and a terminal object, 1), a *universal monomorphism* $1 \xrightarrow{\text{true}} \Omega$ (i.e. for any monomorphism of \mathfrak{E} $A' \xrightarrow{m} A$ there exists a unique "characteristic function" such that the diagram

$$\begin{array}{ccc}
 A' & \xrightarrow{\quad} & 1 \\
 \downarrow m & & \downarrow \text{true} \\
 A & \xrightarrow{x_m} & \Omega
 \end{array}$$

is a pull-back), and for each object its power set Ω^A (this is characterized by the fact that the morphisms $X \rightarrow \Omega^A$ are precisely the subobjects of $X \times A$, in particular its global sections $1 \rightarrow \Omega^A$ are the subobjects of A). The most common examples of topos are the category of sets, \mathfrak{S} , categories of functors $\mathfrak{S}(C^{\text{op}})$ for any small category C , and categories of sheaves on topological spaces. Details about these can be found in [1] or [2]. One of the main consequences of the axioms is that Ω (the "truth table" object) is a Heyting algebra object. (A Heyting algebra is a lattice with "pseudocomplements". The open set lattice of a topological space is a typical example.) Roughly speaking a topos could be thought of as a model for intuitionistic set theory (subobjects do not have honest complements).

In this setting the axiom of choice reads:

AC: Every epimorphism has a section.

Theorem. *Any coequalizer of two nonintersecting monomorphisms has a section iff in \mathfrak{E} subobjects have complements.*

Proof. Let $A' \xrightarrow{m} A$ be a monomorphism in \mathfrak{E} and construct the fol-

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lowing coequalizer diagram:

$$(1) \quad A' \begin{matrix} \xrightarrow{mi_1} \\ \xleftarrow{mi_2} \end{matrix} A + A \xrightarrow{p} A + A$$

in which by hypothesis p has a splitting. $A + A$
 A' can also be obtained from the push-out diagram:

$$(2) \quad \begin{array}{ccc} A' & \xrightarrow{m} & A \\ m \downarrow & & \downarrow j_1 \\ A & \xrightarrow{j_2} & A + A \end{array}$$

From the general theory of topos it follows that j_1 and j_2 are monomorphisms and that (2) is a pull-back (hence an intersection).

Let $A + A \xrightarrow{s} A + A$ be a section of p . The mere existence of such a morphism (see [1] or [2]) forces $A + A$
 A' to be the form $D + E$ where $D = s^{-1}(A + 0)$, $E = s^{-1}(0 + A)$. Similarly $s \cdot j_1$ and $s \cdot j_2$ produce two decompositions of A , namely $A = B_1 + C_1$ and $A = B_2 + C_2$, hence

$$A = B_1 \cap B_2 + B_1 \cap C_2 + C_1 \cap B_2 + C_1 \cap C_2.$$

Thus the diagram (2) becomes

$$(3) \quad \begin{array}{ccc} A' & \xrightarrow{m} & B_1 + C_1 \\ m \downarrow & & \downarrow k_1 + l_1 \\ B_2 + C_2 & \xrightarrow{k_2 + l_2} & D + E \end{array}$$

therefore $A' = B_1 \cap B_2 + C_1 \cap C_2$ and obviously has a complement $\neg A' = B_1 \cap C_2 + C_1 \cap B_2$.

Conversely, if

$$A' \begin{matrix} \xrightarrow{m} \\ \xleftarrow{n} \end{matrix} A$$

are such that $m \cap n = 0$ then $A = m + \neg m$ and $A = n + \neg n$, hence

$$A = m \cap \neg n + \neg m \cap n + \neg m \cap \neg n = m + n + \neg m \cap \neg n$$

i.e. $A = A' + A' + B$ for some object B . But then the coequalizer of m and n is

$$A' \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{i_2} \end{array} A' + A' + B \xrightarrow{\nabla+B} A' + B$$

which is obviously split by

$$A' + B \xrightarrow{i_1+B} A' + A' + B.$$

Corollary. *AC implies that every subobject has a complement.*

Corollary. *If in $\text{Sh}(T)$ epimorphisms (or even only coequalizers of non-intersecting monomorphisms) split then every open set in T is clopen and T is the disjoint union of sets with the indiscrete topology.*

The present version of the proof is the result of several discussions with M. Barr in which he pointed out that the amount of topos language can be reduced to a minimum.

REFERENCES

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2. M. Tierney, *Axiomatic sheaf theory*, Some Constructions and Applications in Categories and Commutative Algebra (P. Salmon, editor), Edizioni Cremonese, Roma, 1973.

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