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TRANSCENDENCE IN OBJECTIVE NUMBER THEORY

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To Bill Lawvere

Introduction

A few years ago, when Bill Lawvere and I were first teaching the elementary course which became our book [2], we needed simple examples of 'categories of cohesion', categories which exhibited the general features of categories of topological or algebraic or analytic spaces, but were simple enough to be grasped by beginners. We chose to concentrate on two presheaf categories, finite dynamical systems and finite directed graphs, and used these to illustrate universal constructions, such as products, coproducts, and function spaces.

Of course we showed the students that these categories share many of the general features of the category of finite sets, so that the arithmetic of addition, multiplication, and exponentiation of isomorphism classes of objects in such a category could be regarded as a natural analog of the arithmetic of natural numbers. While we were thus able to illustrate the categorical roots of the basic identities satisfied by these arithmetic operations, the matter was not pursued very far; hence it seemed to us that some light might be shed by investigating this 'objective number theory' more fully.

Most of the progress came by first studying the example of the category *Graph* of finite directed graphs, in which an object is a pair of finite sets called 'arrows' and 'dots', together with two maps, called 'source' and 'target' from arrows to dots. In my talk in Perugia, I described how this led to a general theory of addition and multiplication, in any extensive category with finite products, in which every infinite chain of retracts terminates. (A category $\mathbf{E}$ is *extensive* iff it has finite coproducts and for each pair A, B of objects, the obvious functor $\mathbf{E}/A \times \mathbf{E}/B \to \mathbf{E}/A + B$ is an equivalence.) This includes basic examples from algebraic/analytic geometry, e.g. the opposite of the category of commutative Noetherian rings.

Since I spoke about the same topic later in Montreal, also at an occasion honoring Bill, and an account of that talk will appear elsewhere [3], it seems preferable to describe here some more recent results, involving exponentiation.

This subject is still in preliminary form, as so far only the special case of finite graphs has been studied, and I do not yet know whether this will lead to an understanding of more general categories. Still, even this example may be of interest, since the analogous questions in classical number theory have been studied intensively for several centuries, but have eluded all efforts to reach a complete solution.

Hilbert's seventh problem

In 1741, Euler proposed that $2^{\sqrt{2}}$ is transcendental, that is, not a root of any non-zero polynomial with integer coefficients. This was rather bold, for it was another hundred years before Liouville was able to prove the existence of transcendental numbers, by showing that $\sum_{n=0}^{\infty} 10^{-n!}$ is transcendental. (Cantor's observation that there are more real numbers than algebraic numbers was still well in the future.) Considerable progress was made by Lindemann, Weierstrass, and others, but at the turn of the century Euler's original problem was still untouched, and Hilbert included it as the seventh of the problems for our century to attack. Gelfond and Schneider finally showed that Euler was correct, and Baker and many others made further progress since, but there is still a lot that we do not know about algebraic powers of rational numbers.

The most optimistic conjecture about the algebraic behavior of the complex exponential function, which suggested itself to the author in his student days, is that if $a_1, \ldots, a_n$ are complex numbers linearly independent over the rationals, then among the $2n$ numbers a_i, e^{a_i}, some n are algebraically independent, i.e. do not constitute a zero of any polynomial in n variables with integer coefficients. Current methods seem still to be far from proving this, or even from proving one of its consequences relevant to our story, namely that if $p_1, \ldots, p_m$ are distinct primes and $1, a_1, \ldots, a_n$ are algebraic numbers linearly independent over the rationals, then the mn numbers $p_i^{a_j}$ form an algebraically independent list.

Rational and antirational algebraic graphs

We say that a finite graph G is *algebraic* if there exist distinct polynomials $p(T), q(T)$, with natural number coefficients, such that $p(G)$ is isomorphic to $q(G)$. By the general theory described in [3], G is algebraic if and only if it is *separable* (the diagonal $G \to G \times G$ is a summand) iff the spectrum of G (the set of cardinalities $\#Graph(C, G)$, C connected) is finite; and then $p(G)$ is isomorphic to $q(G)$ iff $p(T)$ and $q(T)$ agree on the spectrum of G. Since in a presheaf category an object is separable iff as a functor it acts monomorphically, it follows that an algebraic graph is uniquely a sum of connected graphs from two infinite families: *arrow-strings* A_n for $n \geq 0$, with $n+1$ dots and n arrows arranged in a line head to tail, and *cycles* C_n for $n \geq 1$, with n dots and n arrows arranged in a circle head to tail, $C_1 = 1$ being the terminal object. (If a graph is not algebraic we say it is *transcendental.*)

In an extensive category, if X and $X \times Y$ are nonzero 'natural numbers' (sums of 1's) then so is Y, thus there is no graph 'two-thirds', etc. Hence we decide to use the term 'rational graph' in the sense suggested by Euler's observation:

Definition: An algebraic graph E is *rational* iff for all B, B algebraic implies B^E algebraic.

An easy general theorem shows that for any E, separable or not, the following are equivalent:

(1) for all B, B separable implies B^E separable;

(2) 2^E is separable;

(3) the map *true*: $1 \to 2^E$ is an isolated point (i.e. a summand);

(4) multiplication by E preserves density of maps, where a map $X \to Y$ is called 'dense' if it factors through no proper summand of Y.

Specializing to graphs, we see that the rational graphs are exactly sums of cycles, so that every algebraic graph is uniquely a sum of a rational graph and an *antirational* graph, i.e. a sum of arrow-strings A_n.

Algebraic independence of rational graphs to antirational powers

By the Isbell-Pultr theorem [1], two finite graphs X and Y are isomorphic provided that for each finite graph C, the cardinalities of the sets $Graph(C, X)$ and $Graph(C, Y)$ are equal, and it clearly suffices to restrict to connected graphs C. From this it follows that if R is rational, with n dots, and A is antirational then R^A is isomorphic to n^A. To see this, note that maps $C \to R^A$ correspond to maps $C \times A \to R$, and that there is a map from $C \times A$ to some A_n, from which it follows that the number of maps from any connected component of $C \times A$ to R is equal to the number of dots of R, since R is a sum of cycles.

Thus to understand rational graphs to antirational powers, it suffices to understand natural graphs (sums of 1's) to antirational powers, and these are governed by the following theorem:

Theorem I: The family p^{A_n}, with p ranging over primes, n over natural numbers, is algebraically independent.

That is, if $G_1, \ldots, G_k$ are distinct graphs in that family, and p, q are polynomials with natural number coefficients in k variables such that $p(G_1, \ldots, G_k) \cong q(G_1, \ldots, G_k)$, then $p = q$. The theorem is proved by first showing that each monomial, or product of powers of graphs in the family, is connected; this reduces the problem to the case where each of p and q is a single monomial. A further application of Isbell-Pultr, together with the uniqueness of prime factorization of natural numbers, then reduces us to the case of a single prime p; we must show that if $p^A \cong p^{A'}$ then $A \cong A'$. Because p has a *point* (a map from 1), we have multiplicative cancellation and can assume that A involves A_n while A' involves only those A_i with $i < n$. Then we observe that in $p^{A'}$, between any two points there are directed paths shorter than those possible in p^A, giving the desired contradiction.

Allowing algebraic coefficients

At first sight, it might seem that the complete story of the algebra of rational graphs to algebraic powers has been reduced by Theorem 1 to the algebra of algebraic graphs, since in the rig generated by algebraic graphs and algebraic powers of rational graphs we have succeeded in reducing every element to a sum of terms of type KM where K is algebraic and M is a product of graphs p^{A_n}. Since the graphs p^{A_n} are algebraically independent, what remains to be done? The difficulty is that experience with classical number theory is misleading, because of the failure here of multiplicative cancellation. If graphs are algebraically independent over the natural numbers, we cannot conclude that they are algebraically independent over the rig of algebraic graphs. The simplest example is $A_0 2^{A_0} \cong 2A_0$; even though 2^{A_0} is transcendental, we can multiply it by the non-zero algebraic graph A_0 (an isolated dot) and get an algebraic result. Fortunately, it turns out that this simple example easily generalizes to give a normal form.

Theorem 2: Any element of the rig generated by algebraic graphs and algebraic powers of rational graphs is uniquely a sum of connected elements, each of which is in turn uniquely a finite product $C\Pi p^{S_p}$, where C is connected and algebraic, each S_p is antirational and if C is A_n, then the exponents S_p involve only those A_i with $i < n$.

The proof follows the pattern of Theorem 1; more details (and hopefully more general results) will appear in a book on objective number theory which Bill and I are in the process of writing.

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