4

Tools for the Advancement of Objective Logic: Closed Categories and Toposes

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The thesis is that the explicit adequate development of the science of knowing will require the use of the mathematical theory of categories. Even within mathematical experience, only that theory has approximated a *particular* model of the general, sufficient as a foundation for a *general* account of all particulars. Arising 50 years ago from the needs of geometry, category theory has developed such notions as adjoint functor, topos, fibration, closed category, 2-category, etc., in order to provide:

(1) A guide to the complex, but very non-arbitrary constructions of the concepts and their interactions which grow out of the study of space and quantity.

It was only the relentless adherence to the needs of that basic subject that made category theory so well-determined yet powerful. When some schools of category theory have gone astray, it has usually been due either to neglecting too long that specific goal of studying space and quantity better, or to ossifying some partial determination of what space and quantity are. If we replace "space and quantity" in (1) above by "any serious object of study," then (1) becomes my working definition of *objective logic*. Of course, when taken in a philosophically proper sense, space and quantity do pervade any serious field of study. Category theory has also objectified as a special case:

(2) The subjective logic of inference between statements. Here statements are of interest only for their potential to describe the objects which concretize the concepts; here by describing, we mean both commenting on the objects constructed and indicating desiderata for their construction.

Specifically, we need, for example, a mathematical model of the following philosophical position. Within thinking:

(a) Subjective logic is a part of objective logic, which also reflects and partly guides construction in the latter.
(b) Thinking itself is a part of being, which reflects being and guides our action on it.
(c) One of the many aspects of being is (b) itself, which is therefore reflected to manifest itself as (a).
(d) Considered as a process within being, (a) is a central feature of thinking which a *science* of thinking must address.

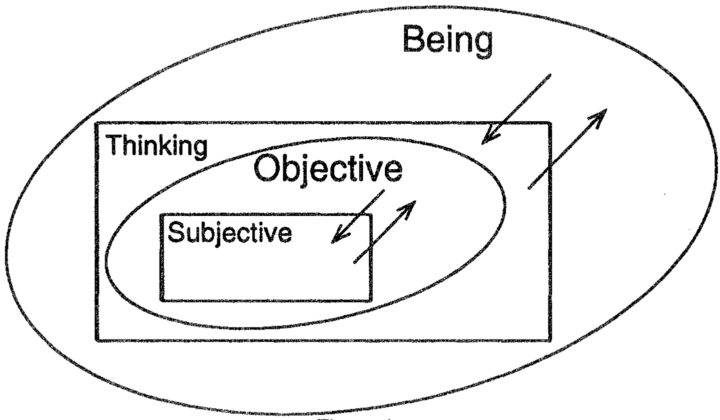


Figure 1.

I believe that there is a second central feature of thinking: individuals think, but so do clans, "schools of thought," professions, social classes, nations, etc.; thinking takes place in brains, but also in schools, newspapers, meeting houses, etc. – i.e., in material institutions which people have created for the purpose of mutually transforming individual thinking and collective thinking into each other.

An explicit philosophy, dealing with both the mutual transformation of subjective logic and objective logic, as well as the mutual transformation of individual thinking and collective thinking, could be very helpful (for example, when reforming the schools) and adequate mathematical models will be an essential basis of clarity.

1. Categorical Refinement of a Hegelian Principle

A very generally occurring specific kind of process is the following. A category $\mathcal{B}$ of *particulars* is presented to thought and measured against an abstract distillation $\mathcal{V}$ of previous concrete thoughts. There results an *abstract general* $\mathcal{A}$ partly expressing the essence of $\mathcal{B}$ but permitting calculation and speculation within itself. The acquisition of $\mathcal{A}$ permits the construction of the *concrete general* $\mathcal{C}$, a category pictured as containing both all the possible objects of essence $\mathcal{A}$ as well as all the $\mathcal{A}$-respecting transformations and comparisons between these possible objects. At least as important as the calculation in $\mathcal{A}$ is our power to investigate by pure thought the objects in $\mathcal{C}$ (recording perhaps some of the results in $\mathcal{A}$) and thus to create plans. In particular, there is a functor $\mathcal{B} \longrightarrow \mathcal{C}$ reflecting (partly) each of the original particulars as one among the $\mathcal{A}$-possibles.

Mathematical experience has shown that the above description can be made even more precise and productive. In order that the measuring can compare $\mathcal{B}$ and $\mathcal{V}$, $\mathcal{V}$ must also be a category. While a theory $\mathcal{A}$ *may* have (within the abstract general) a useful subjective presentation $\mathcal{A}' \longrightarrow \mathcal{A}$ by means of "primitives" and "axioms," its objective result, the concrete general $\mathcal{C}$, depends only on $\mathcal{A}$ itself as the category presented. The kind $\mathcal{D}$ of category that $\mathcal{A}$ is desired to be, must be a kind in which $\mathcal{V}$ participates too. Then an explicit model of the process can be given as follows:

$$\mathcal{A} \subseteq \text{Hom}(\mathcal{B}, \mathcal{V})$$

a category of functors from $\mathcal{B}$ to $\mathcal{V}$ and all natural transformations between them. This is the idea of "natural structure"[1] (distilled in my 1963 doctoral thesis from examples in algebraic topology; there 'structure' is taken in the sense of an abstract general). This $\mathcal{A}$ typically specifies not only the theorems true for all objects in $\mathcal{B}$, but also the kind of formulas appropriate to $\mathcal{B}$ among which the theorems are found; externally specified primitives are not presupposed. Then:

$$\mathcal{C} = \mathcal{D}\text{Hom}\,(\mathcal{A}, \mathcal{V})$$

the category of $\mathcal{D}$-preserving structures of kind $\mathcal{A}$ valued in $\mathcal{V}$. Here 'structure' is taken in its concrete general aspect. Thus, relative to $\mathcal{D}$ and $\mathcal{V}$, the comparison functor $\mathcal{B} \longrightarrow \mathcal{C} = \mathcal{B}^{**}$ is seen as an instance of the standard Fourier map to the double dual, familiar from the work of Stone, Pontrjagin and others in the 1930s that made explicit the relation:

$$\text{D}^*\text{-spaces} \longleftrightarrow (\text{D-algebras of variable quantities})^{\text{op}}$$

as an adjoint pair of functors with constant quantity V taken as dualizing object/space. There are many kinds of spaces, for we can take D = Boolean (with V = 2) or continuous or smooth or combinatorial, etc.; that is, D functions as an abstract general. These all arose from looking at particular spaces and their relationships. Similarly, the doctrine $\mathcal{D}$ can be not only the notion of finite cartesian products (as in universal algebra), but also of Barr regular (as in positive logic, etc.), and the concrete generals $\mathcal{C}$ which arise belong to $\mathcal{D}^*$-Cat where $\mathcal{D}^*$ is a dual doctrine, as the work of Makkai and others has richly demonstrated. Change of doctrines $\mathcal{D}' \longrightarrow \mathcal{D}$, with its functorially induced changes in the rest, is as much an everyday tool as is interpretation $\mathcal{A}' \longrightarrow \mathcal{A}$ of theories or morphisms $M \longrightarrow M'$ of $\mathcal{A}$-structures in $\mathcal{C}$, since all these have long been recognized to take place in categories, with all which that implies. For example, an interpretation $\mathcal{A}' \longrightarrow \mathcal{A}$ of abstract generals functorially induces a functor $\mathcal{C} \longrightarrow \mathcal{C}'$ in the opposite direction which compares the corresponding concrete generals. For example, if my previous experience $\mathcal{B}$ with cats has led to a theory $\mathcal{A}$ which includes that they have long tails, dropping that axiom leads to an $\mathcal{A}'$ and a resulting $\mathcal{C}'$ which would provide for the possibility of the broadening of my experience when I meet a Manx.

While subjective logic in the narrow sense might take $\mathcal{V}$ just as truth-values, a better example is the category of abstract sets and arbitrary mappings (but for Lie theory, e.g., it might be appropriate to take something much richer, such as the category of smooth spaces, as the distillation of knowledge presumed to be acquired). The category $\mathcal{V}$ of sets obeys various rich doctrines as alluded to above, but it is already quite fruitful just to let $\mathcal{D}$ recognize merely that $\mathcal{V}$ is a category. The possible $\mathcal{D}^*$'s are then abstract theories of sets whose particular concretes include not only the constant $\mathcal{V}$, but also the categories of variable and cohesive "sets" which may arise, for example, as categories $\mathcal{B}$ of particulars to be measured. Measurement of a given $\mathcal{B}$ might reveal, for example, a pair of functors:

$$\mathcal{B} \underset{P}{\overset{A}{\rightrightarrows}} \mathcal{V}$$

assigning sets to each particular, and measurement might even reveal a triple of transformations:

$$A \underset{t}{\overset{s}{\rightrightarrows}} P, \qquad r : P \longrightarrow A$$

which are natural (i.e., homogeneous with respect to all particular comparisons $B \longrightarrow B'$ in $\mathcal{B}$) and satisfy, for example, $sr = 1_P = tr$. An initial version of $\mathcal{A}$ would be just the *finite* category:

defined by the same equations. Then $\mathcal{C} = \mathcal{D}\mathrm{Hom}(\mathcal{A}, \mathcal{V})$ is the infinitely rich category of concrete reflexive graphs whose scientific interest and technological utility will probably never be exhausted; the functor $\mathcal{B} \longrightarrow \mathcal{C}$ reveals that each of our given particulars B "is" at least a reflexive graph whose points P(B) are provided by $\mathcal{B}$ also with a set A(B) of arrow connections and with maps s(B), t(B), r(B) specifying which points are the source and target of given arrows and which arrow is the null loop at each point.

2. Deepening, Metrics, Unity and Identity of Adjoint Opposites

In the above example I have used the pregnant observation of Macnamara and the Reyes that the logical interpretation of the word 'is' should be just a *map* in an appropriate category, *not* necessarily an inclusion map: to do otherwise is a cheap source of paradoxes. A related observation concerns the interpretation of 'and': it is well known that in a category $\mathcal{A}$, categorical product is the right adjoint of the diagonal functor $\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \times \mathcal{A} \xrightarrow{\times} \mathcal{A}$ (where $\mathcal{A} \times \mathcal{A}$ is the product in $\mathcal{D}$-Cat), and that if $\mathcal{A}$ consists only of inclusions (i.e., $\mathcal{A}$ is a poset as in propositional logic) then product reduces to conjunction. It is even exploited in proof theory that for any category $\mathcal{A}$, the poset reflection $\mathcal{A} \xrightarrow{\exists} \mathcal{A}_0$ maps products to conjunctions. However, as pointed out by Schanuel, the most direct interpretation of everyday 'and' is very often product in a non-poset, such as 'the position of the bird is determined by its altitude *and* the point on the earth beneath it'.

As posets often need to be deepened to categories to accurately reflect the content of thought, so should inverses, in the sense of group theory, often be replaced by adjoints. Adjoints retain the virtue of being uniquely determined reversal attempts, and very often exist when inverses do not.

Great mathematical philosophers, such as Grassmann 150 years ago, recognized the distinction between synthetic operations such as addition and the corresponding analytic operations such as subtraction, but the nature of the correspondence cannot be characterized in terms of inverses, even for the posetal case of non-negative real numbers, where the truncated difference is indeed the adjoint:

$$\frac{x + a \geq y}{x \geq y \,\Delta\, a}$$

This is the invertible rule which guides our calculation with bank accounts, etc., $y \,\Delta\, a$ being always non-negative. In that example we could recover the group property by passing to a larger system, but at the loss of the useful criterion $a \geq b$ iff $b \,\Delta\, a = 0$. However, if + denotes an *idempotent* operation such as subjective disjunction of statements or objective union of closed regions in space, then there is no group enveloper, but the analytic adjoint 'yes..., but not...' operation may exist. This adjointness is all that is needed (together with the usual lattice axioms, which are also just adjointnesses) to derive the properties of the topologically crucial boundary operation:

$$\text{boundary (A)} \underset{\text{def}}{=} \text{A and (yes but not A)}$$

Sometimes a given functor has both left and right adjoints. For example, the inclusion of truth values $V_0 \longrightarrow V_\infty$ into distance values, wherein true is 0 ("right on") and false is ∞ ("making it true is prohibitively expensive"), has both left and right adjoint retractions, one (feasibility) whereby every finite distance is true and the other (frugality) whereby every non-zero distance is too much to be truly necessary and hence false. Now, both V_0, V_∞ are closed categories and hence serve, in a way related to but different from the ways discussed in section 1, as abstract generals: the corresponding concrete generals are the categories of posets and of metric spaces (not necessarily symmetric, with distance-decreasing maps) respectively. The trio of little adjoints covariantly induces bigger ones: a poset is a special metric space in which all distances are either 0 or ∞ but the points of a metric space are ordered in two extreme ways, frugality saying $x \longrightarrow y$ iff it costs nothing, but feasibility permitting $x \overset{\bullet}{\longrightarrow} y$ iff the cost is finite. (There are many precise determinations of the category V_∞ of distances; for example, some have ∞ = one trillion, so finite just means "under budget.") This feasibility relation induces an equivalence relation on points (and on maps) of metric spaces, two points x, y being in the same component iff the distances in both directions are finite; this decomposition, in spite of its purely logical character, turns out to be intimately related to the geometrical notion of *rotations*, for these are essentially the only automorphisms of metric spaces which retain any activeness after this components functor has neglected all those which move all points by at most a bounded distance.

As another example of how the theory of closed categories can explicitly guide the study of metric spaces, note that the family of intervals [0,d] in V_∞ provides a special family of metric spaces. Any small family of objects in a complete category raises two questions: are the functions (or properties), i.e., the maps in the category whose codomains are in the family, coadequate to determine every object in the category? And are the paths (or probes), i.e., the maps in the category whose domains are in the family, adequate to determine every object? Isbell in 1960 showed the importance of these questions. A fundamental adjoint construction by Kan (1958) has given rise to a measure of the failure of adequacy by the so-called adequacy comonad $\Gamma X \xrightarrow{e} X$, where Γ is an endofunctor of the big category giving for every object X the best approximation to it which can be reconstructed from the results of probing it with paths parameterized by objects in the given small family, and where e is the natural transformation directly comparing the approximation with the X being investigated (so adequacy holds if e is an isomorphism). In our example, where the big category is the concrete general of all metric spaces determined by the closed category V_∞ of distances and where the small family is the family of intervals [0,d], for d finite from V_∞ itself, the path-approximation ΓX to X turns out to be the *geodesic remetrization* of X; for example, if X is the surface of the earth metrized by its obvious embedding in the earth itself, then $\Gamma X \xrightarrow{e} X$ is the distance decreasing map from the same surface metrized by minimum-distance paths which stay on the surface.

Conversely, many notions suggested by the study of metric spaces, such as radius, engulfing, etc., have precise analogues for concrete generals enriched in any (even non-posetal) closed categories V.

Sometimes (opposite to the case discussed above) a functor p which has both adjoints is itself a common retraction for both of its adjoints. Any such functor is a precise realization of the allegedly nebulous notion *unity-and-identity-of-opposites*. For the two adjoints are then two inclusions of subcategories, united in the domain of p, opposite in the precise sense given by the adjointness itself, yet both identical in themselves with the codomain of p; moreover, for each object X in the domain of p there is a well-defined "interval" $L_p(pX) \longrightarrow X \longrightarrow R_p(pX)$ with endpoints in the two opposite subcategories, within which X must lie, but determined only by the partial knowledge pX about X. As a very simple example, consider the case where p is any surjective, order-preserving map of finite, totally ordered sets. A more general image is that the domain of p is structured as a "cylinder" with two identical ends which are the two subcategories and with threads running through it from each object on the left end to the corresponding object on the right end; all the objects with a given p-value are on one thread, and certain of the comparison maps between them point along the thread. Functoriality gives ways of passing between threads. However, in many examples even the approximating interval around X is not a poset; also special care must be taken not to assume that the two opposite subcategories are disjoint, for the two ends of the cylinder often touch in certain points.

From examples previously discussed, we can assemble an example of UIAO (unity and identity of adjoint opposites). Reinterpret the distance category V_∞ as a category of times and consider the functor category $\mathcal{V}^{V_\infty^{op}}$: its objects are systems of sets X_t (for t in V_∞) of possible states equipped with a definite law of becoming $X_t \longrightarrow X_s$ for $s \geq t$ which satisfies the evident transitivity conditions, and its maps $X \longrightarrow Y$ are systems $X_t \longrightarrow Y_t$ of $\mathcal{V}$-maps which compatibly compare the two respective laws of becoming. There are two opposite subcategories of very special systems: in one the change of states $X_0 \longrightarrow$ can be arbitrary, but $X_t \longrightarrow X_s$ for $s \geq t > 0$ is always an identity map so that "everything happens right at the beginning"; in the other $\longrightarrow X_\infty$ can be arbitrary but $X_t \longrightarrow X_s$ for $\infty > s \geq t$ is always an identity so that "everything happens right at the end." Suppose, for an arbitrary X, that we know only $Y = p(X) = [X_0 \longrightarrow X_\infty]$, a pair of sets structured by a single map, but that we are as yet ignorant of the details of what happened in between. Then we can start further investigation from two approximations LY $\longrightarrow X \longrightarrow$ RY where LY, RY are in the two opposite subcategories and the arrows are maps in $\mathcal{V}^{V_\infty^{op}}$. The two subcategories are in themselves isomorphic to $\mathcal{V}^{V_0^{op}}$: (which has as objects single maps $Y_{true} \longrightarrow Y_{false}$ of sets) and the functor p which adjointly unites them is induced by our previous inclusion $V_0 \longrightarrow V_\infty$. Here the categories V_0, V_∞ are playing a different role as abstract generals, more akin to the one discussed in section 1: they specify kinds of models of becoming.

Also a concrete general $\mathcal{U}$ of being, whose abstract general specifies a kind of cohesiveness or unity of being, may participate in UIAO's. To take an example, Cantor's abstraction process $\mathcal{B} \longrightarrow \mathcal{V}$, assigning to particular cohesive spaces their abstract sets of points, extend (along $\mathcal{B} \longrightarrow \mathcal{B}^{**} \underset{\text{def}}{=} \mathcal{U}$) to a functor p whose left and right adjoint inclusions are the subcategory $\mathcal{V} \longrightarrow \mathcal{U}$ of discrete spaces and $\mathcal{V} \longrightarrow \mathcal{U}$ of codiscrete spaces respectively. The contradictory properties of an abstract set, namely that its points are completely distinguished yet indistinguishable by any clear property, are resolved into two opposite spaces with the same points: the discrete one in which no connected motion exists to blur them and the codiscrete one in which every point is instantly blurred into every other one; though there is the canonical LS $\longrightarrow$ RS map of discrete to codiscrete which induces an isomorphism on S upon applying p, typically the only maps ("clear properties") from a codiscrete to a discrete space are constant. The relevant abstract generals say that a map in $\mathcal{U}$ should preserve cohesion, but a discrete space D has zero cohesion to preserve, so maps D $\longrightarrow$ X are arbitrary on points; dually, a codiscrete space C has in a vacuous way infinite cohesion, so again maps X $\longrightarrow$ C are arbitrary on points; these remarks explain the two adjointnesses. The notion of connected movement in a space X (discrete or not) can be usefully modelled by maps T $\longrightarrow$ X in $\mathcal{U}$ where T is a connected space; meaning that $\pi(T)$ is one point where π is the further left adjoint $\mathcal{U} \longrightarrow \mathcal{V}$ to the discrete inclusion and called 'the set of connected components of ___'.

Objective logic must recognize the quality of dimensionality in spaces and construct quantity-types for measuring it. I have outlined elsewhere (Lawvere 1991b) a program for doing this in terms of intermediate UIAO's $\mathcal{U} \longrightarrow \mathcal{U}_n \longrightarrow \mathcal{V}$ related by a left-to-right "crossover" determination of Hegel's jump idea. I showed its correctness in some important examples, and in particular defined dimension one as the lowest UIAO between $\mathcal{U}$ and $\mathcal{V}$ for which $\pi(L_nX) = \pi X$ for all spaces X in $\mathcal{U}$, that is, all connecting which can be done can be achieved with one-dimensional paths. However, here I will approach dimension more directly in connection with a combinatorial example.

3. Realization of Plans

One should not despise the codiscrete spaces $\mathcal{V} \longrightarrow \mathcal{U}$, for their very contrast with the discrete ones permits them to be seen as connected "blobs" of various dimensions which can be glued into quite arbitrary combinatorial plans. What such a plan envisages is a real space (or building, etc.) to be constructed, using previously-acquired components: a two-point blob is imagined as a continuous one-dimensional segment with two endpoints, a three-point blob as a continuous two-dimensional triangle with three edges, a four-point blob as a three-dimensional tetrahedron, etc. The simplest example of a non-codiscrete space obtained by gluing together blobs is a *one* - dimensional triangle consisting only of the edges. Certain considerations would require that even the one-dimensional segment be recognized to have an infinite-dimensional microstructure, but whatever may be the precise nature of these real components to be used in carrying out the plans, their specification can be considered to be a standard additionally given functor $\mathcal{V} \longrightarrow \mathcal{U}$. Both the gluing of codiscrete pieces to see plans and also the gluing of standard pieces to carry them out, are realized as colimits in the category $\mathcal{U}$. The standard-space functor $\mathcal{V} \longrightarrow \mathcal{U}$ induces an adjoint pair:

$$\mathcal{U} \rightleftarrows \mathcal{V}^{\mathcal{V}^{\text{op}}}$$

where each space is considered as the structured ensemble of possible probings of it by standard spaces, and the left adjoint return "realizes" any conceivable $\mathcal{V}^{\text{op}} \xrightarrow{P} \mathcal{V}$ by using it as a scheme to glue together standard pieces. But wait.

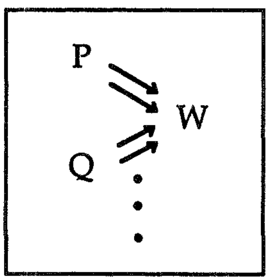
As the relation between the subjective and objective can be reflected into the subjective, for example, as the relation between presentations by axioms and invariant theories within the abstract general, so can it be reflected into the objective, as in our present description, using concrete generals of spaces and of combinatorial plans, etc. But there is a difference. Speculation costs little, and it seems that calculation can go on indefinitely; thus perhaps countable infinity is not such a bad idealized model of those aspects of subjectivity. By contrast, a plan which can be carried out must be finite.

Thus we introduce a reduction $\mathcal{V}^{\mathcal{V}^{\text{op}}} \longrightarrow \mathcal{U}'$ of our dreamy scale of plans, in order that the composite $\mathcal{U}' \xrightarrow{\text{left adj.}} \mathcal{V}^{\mathcal{V}^{\text{op}}} \longrightarrow \mathcal{U}$ comes closer to modelling constructions which can really be done. For example, all homotopy types of spaces arise from the standard choice $\mathcal{U}' = \mathcal{V}^{\mathcal{K}_0^{\text{op}}}$, where $\mathcal{K}_0 \longrightarrow \mathcal{V}$ (inducing the reduction) is just the inclusion of the category of finite sets. In general, such a composite $\mathcal{U}' \longrightarrow \mathcal{U}$ reflects, within objective logic, the subjective-objective opposition itself; it is the concrete result of a comparison between two abstract theories of space, a combinatorial one and a continuous one.

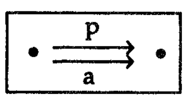
4. Detailed Models of Becoming

Sometimes the objects in being to be constructed are actually in the realm of thinking (in the broad material sense we are giving to thinking). For example, a book may be constructed from an outline. There are also shorthand notes for delivering a speech (wherein the symbols in the combinatorial plan are to be realized by syllables of sound) and librettos for operas with musical symbols, etc. If we regard the desired product to be an audio tape of the speech or a video tape of the opera, then the geometric sort of realization discussed in section 3 applies. However, in those cases the plan is primarily a plan for a *performance*, a kind of becoming that we must also effectuate when we actually *carry out* the construction of a house or of a machine. The study of the categories and functors appropriate to this problem sometimes goes under the name of "control theory," but these categories have also specific features in common with those arising in continuum mechanics in connection with the tempering of metals, etc. They involve taking into the already acquired $\mathcal{V}$ (hence into the abstract generals $\mathcal{A}$ which arise) a model of time which is not merely a poset or group, but is a category which also contains the possible controlling processes as maps. The corresponding concrete generals are categories whose objects consist of systems of states evolving in a (partly) controlled manner.

Apart from generalities about evolution of states (controlled or not), there are two further important features which make possible specific mathematical treatment. First, states are states *of* a body (often the relation is contravariant) and bodies have parts. The part-whole relation is not merely a poset, but rather each part is included in the whole in two senses, since each part is both itself *and* its relationship; we can call these two kinds of arrows the passive and active inclusions. In its purest form such a body is the category B:



equipped with a faithful labelling functor to:



whose sections correspond to the parts. A system of states for B is a functor $B^{op} \xrightarrow{X} \mathcal{V}$ (where $\mathcal{V}$ might be taken as sets or spaces). The passive maps in B then induce projections $X(W) \longrightarrow X(P)$ for each part P, each state of the whole involving in particular a state of P; these taken together induce a map $X(W) \longrightarrow \Pi_{p \neq w} X(P)$ into the product of the state spaces of all the parts. One usually assumes that this passively induced map is an isomorphism or at least an inclusion: the state of the whole is nothing but collectively the states of all its parts. But there are still the maps $X(W) \longrightarrow X(P)$ obtained by applying X to the *active* maps in B;

these help to express how the collective state will *influence* the state of each part P. An appropriate model of *time* in this context is as *another* functor $B^{op} \xrightarrow{T} \mathcal{V}$ in the same category as X. The passively induced maps T(W) ⟶ T(P) are *individually* assumed to be isomorphisms if we want to express that in a motion or development of B as a whole, the times of each part are synchronized. But there are still the *actively*-induced maps T(W) ⟶ T(P) which I will take as expressing the *delay* with which the individual states of P can respond to the influence of the collective states. Then a motion or development of B which follows the specific laws inherent in X and T is just any map T ⟶ X which is homogeneous (or natural) with respect to *each* of the maps in B. In systems amenable to differential equations, the delays are taken as infinitesimal translations, but they may also be the finite delays inherent in physical connecting "wires."

The second important feature (making calculation feasible) of the evolution of bodies with parts is the following. In the thinking of the body politic, democracy begins with the people U whom individual P knows. Similarly, in a fluid body, the immediate influence on the motion of a particle P comes from the particles very close to it, considered, in partial-differential equations, to constitute an infinitesimal neighborhood U of P. Thus, we can deepen our "purest" model B by introducing an intermediate layer of objects U with sub-whole inclusions U ⟶ W, across which some of the passive and active inclusions of the various P's factor. The new category $\hat{B} \supset B$ obtained depends on the precise interlocking of these sub-wholes (essentially a binary relation between P's and U's). If the state functor X previously discussed is the restriction of an X defined on $\hat{B}$, then the *active* influences factor as:

$$X(W) \longrightarrow X(U) \longrightarrow X(P)$$

making explicit the way in which the immediate influence of the state of the whole on P's state depends only on the state of his neighbours (of course including his own state X(U) ⟶ X(P), due to the factored *passive* inclusion P ⟶ U). The product condition placed previously on X becomes over $\hat{B}$ the inverse limit condition X(W) = lim X(U), the well-defined part (of the product over all objects of $\hat{B}$) consisting of logically possible collective states.

If $\mathcal{V}$ is a topos, so are the categories $\mathcal{V}^{B^{op}}$, $\mathcal{V}^{\hat{B}^{op}}$ in which our state and time objects reside and map into each other; there are many objects in these categories which satisfy neither of the two extreme isomorphism conditions. But some of these objects serve, as in other parts of topos theory, to internalize (to the concrete general determined by the abstract body structure B) the logical and other algebraic calculations we may need to do on functions of state and time. There may also be means, available to objective logic, for making precise the idea that the sub-wholes U of W are qualitatively smaller, as is true at least in the case of the partial-differential equations of continuum motion. Namely, certain very special objects D in $\mathcal{V}$ admit use as "fractional exponents," via a functor $(\ )^{1/D}$ right adjoint to the usual function-space functor $(\ )^D$. In the same way, the idea that the time delays in T are "immediate" may be made precise.

Conclusion

Despite some simplifications in the above, needed for rapid description, I hope that I have made clear that there is a great deal of useful precision lying behind my illustrations, and a great deal to be developed on the same basis. Thus I believe to have demonstrated the plausibility of my thesis that category theory will be a necessary tool in the construction of an adequately explicit science of knowing.

Note

1 A simple example of natural structure: Galois observed that the roots of a particular polynomial equation can be permuted, leading to a certain *group* G as abstract general. The corresponding concrete general is the category of all possible permutation representations of G. (It can be shown that each of these possibles actually arises from a suitable system of polynomial equations.) This example is related to another one. Over a particular domain in the plane, there are varialbe quantities; it was observed that these quantities can be added and multiplied, leading to the abstract general D which is the *algebraic theory of commutative rings*. The corresponding concrete general is the category of all possible commutative rings. (Algebraic geometry shows that every one of these possibles, C, is actually the system of all variable quantities over a suitable domain space, whose dimension and quality depends on the ring C.

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