Cambridge University Press
978-0-521-89485-2 - Conceptual Mathematics, 2nd Edition: A First Introduction to Categories
F. William Lawvere and Stephen H. Schanuel
Excerpt
More information

Preview

SESSION 1

Galileo and multiplication of objects

1. Introduction

Our goal in this book is to explore the consequences of a new and fundamental insight about the nature of mathematics which has led to better methods for understanding and using mathematical concepts. While the insight and methods are simple, they are not as familiar as they should be; they will require some effort to master, but you will be rewarded with a clarity of understanding that will be helpful in unravelling the mathematical aspect of any subject matter.

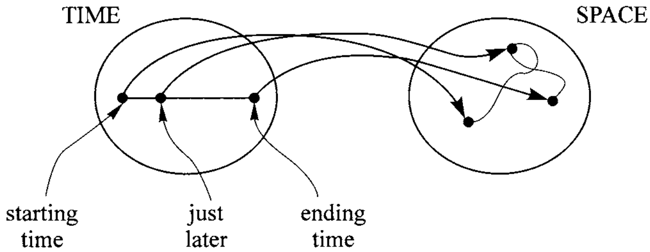
The basic notion which underlies all the others is that of a *category*, a 'mathematical universe'. There are many categories, each appropriate to a particular subject matter, and there are ways to pass from one category to another. We will begin with an informal introduction to the notion and with some examples. The ingredients will be objects, maps, and composition of maps, as we will see.

While this idea, that mathematics involves different categories and their relationships, has been implicit for centuries, it was not until 1945 that Eilenberg and Mac Lane gave *explicit* definitions of the basic notions in their ground-breaking paper 'A general theory of natural equivalences', synthesizing many decades of analysis of the workings of mathematics and the relationships of its parts.

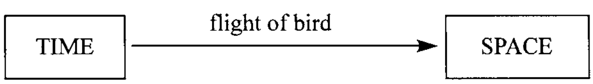
2. Galileo and the flight of a bird

Let's begin with Galileo, four centuries ago, puzzling over the problem of motion. He wished to understand the precise motion of a thrown rock, or of a water jet from a fountain. Everyone has observed the graceful parabolic arcs these follow; but the motion of a rock means more than its track. The motion involves, for each instant, the position of the rock at that instant; to record it requires a motion picture rather than a time exposure. We say the motion is a 'map' (or 'function') from time to space.

The flight of a bird as a map from time to space



Schematically:

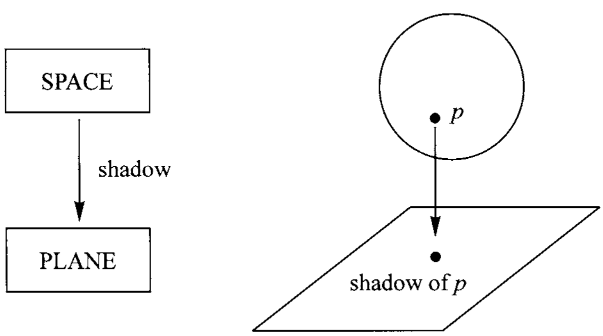


You have no doubt heard the legend; Galileo dropped a heavy weight and a light weight from the leaning tower of Pisa, surprising the onlookers when the weights hit the ground simultaneously. The study of vertical motion, of objects thrown straight up, thrown straight down, or simply dropped, seems too special to shed much light on general motion; the track of a dropped rock is straight, as any child knows. However, the motion of a dropped rock is not quite so simple; it accelerates as it falls, so that the last few feet of its fall takes less time than the first few. Why had Galileo decided to concentrate his attention on this special question of vertical motion? The answer lies in a simple equation:

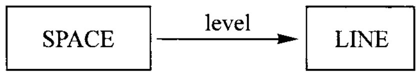
$$\text{SPACE} = \text{PLANE} \times \text{LINE}$$

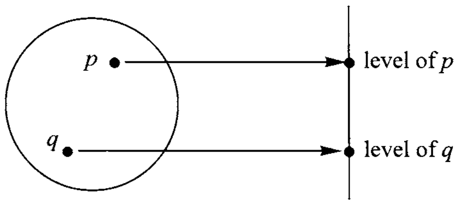
but it requires some explanation!

Two new maps enter the picture. Imagine the sun directly overhead, and for each point in space you'll get a shadow point on the horizontal plane:

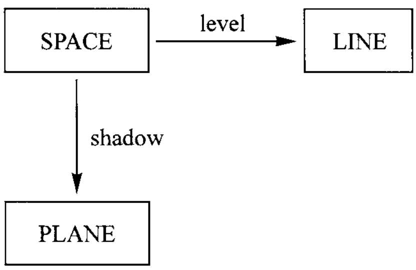


This is one of our two maps: the 'shadow' map from space to the plane. The second map we need is best imagined by thinking of a vertical line, perhaps a pole stuck into the ground. For each point in space there is a corresponding point on the line, the one at the same level as our point in space. Let's call this map 'level':



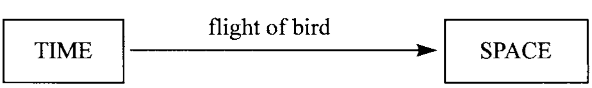


Together, we have:

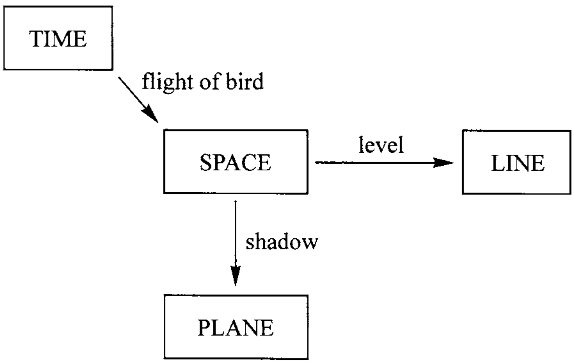


These two maps, 'shadow' and 'level', seem to reduce each problem about space to two simpler problems, one for the plane and one for the line. For instance, if a bird is in our space, and you know only the shadow of the bird and the level of the bird, then you can reconstruct the position of the bird. There is more, though. Suppose you have a motion picture of the bird's shadow as it flies, and a motion picture of its level – perhaps there was a bird-watcher climbing on our line, keeping always level with the bird, and you filmed the watcher. From these two motion pictures you can reconstruct the entire flight of the bird! So not only is a position in space reduced to a position in the plane and one on the line, but also a motion in space is reduced to a motion in the plane and one on the line.

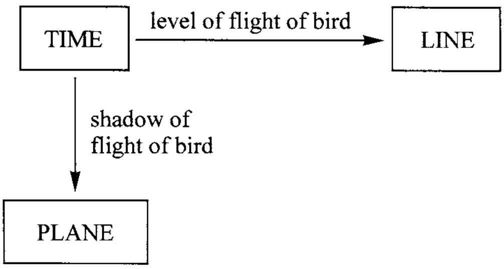
Let's assemble the pieces. From a motion, or flight, of a bird



we get two simpler motions by 'composing' the flight map with the shadow and level maps. From these three maps,



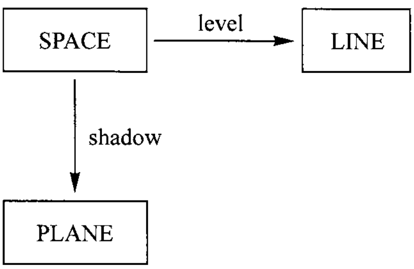
we get these two maps:



and now space has disappeared from the picture.

Galileo's discovery is that from these two simpler motions, in the plane and on the line, he could completely recapture the complicated motion in space. In fact, if the motions of the shadow and the level are 'continuous', so that the shadow does not suddenly disappear from one place and instantaneously reappear in another, the motion of the bird will be continuous too. This discovery enabled Galileo to reduce the study of motion to the special cases of horizontal and vertical motion. It would take us too far from our main point to describe here the beautiful experiments he designed to study these, and what he discovered, but I urge you to read about them.

Does it seem reasonable to express this relationship of space to the plane and the line, given by two maps,

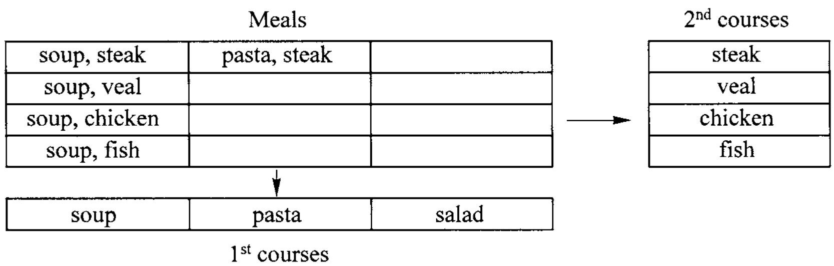


by the equation SPACE = PLANE × LINE? What do these maps have to do with multiplication? It may be helpful to look at some other examples.

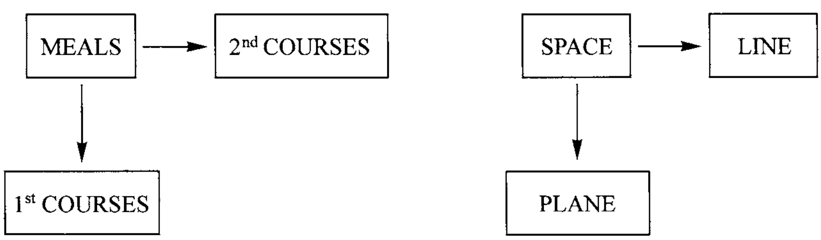
3. Other examples of multiplication of objects

Multiplication often appears in the guise of *independent choices*. Here is an example. Some restaurants have a list of options for the first course and another list for the second course; a 'meal' involves one item from each list. First courses: soup, pasta, salad. Second courses: steak, veal, chicken, fish.

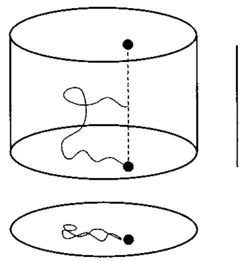
So, one possible 'meal' is: 'soup, then chicken'; but 'veal, then steak' is not allowed. Here is a diagram of the possible meals:



(Fill in the other meals yourself.) Notice the analogy with Galileo's diagram:

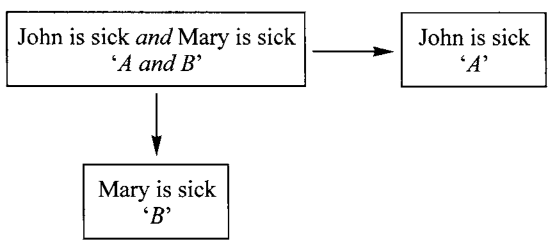


This scheme with three 'objects' and two 'maps' or 'processes' is the right picture of multiplication of objects, and it applies to a surprising variety of situations. The idea of multiplication is the same in all cases. Take for example a segment and a disk from geometry. We can multiply these too, and the result is a cylinder. I am not referring to the fact that the *volume* of the cylinder is obtained by multiplying the area of the disk by the length of the segment. The cylinder *itself* is the product, segment times disk, because again there are two processes or projections that take us from the cylinder to the segment and to the disk, in complete analogy with the previous examples.

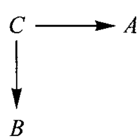


Every point in the cylinder has a corresponding 'level' point on the segment and a corresponding 'shadow' point in the disk, and if you know the shadow and level points, you can find the point in the cylinder to which they correspond. As before, the motion of a fly trapped in the cylinder is determined by the motion of its level point in the segment and the motion of its shadow point in the disk.

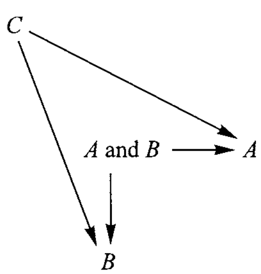
An example from logic will suggest a connection between multiplication and the word '*and*'. From a sentence of the form '*A and B*' (for example, 'John is sick *and* Mary is sick') we can deduce *A* and we can deduce *B*:



But more than that: to deduce the single sentence 'John is sick and Mary is sick' from some other sentence *C* is the same as deducing each of the two sentences from *C*. In other words, the two deductions

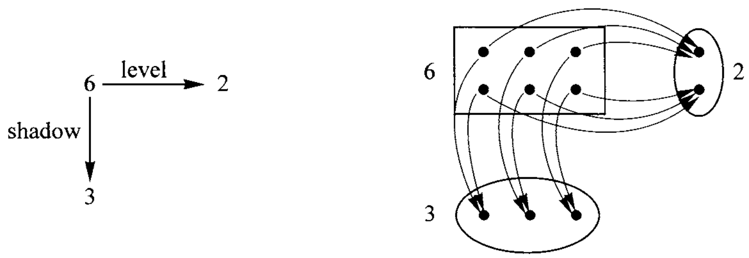


amount to one deduction $C \longrightarrow (A \textit{ and } B)$. Compare this diagram



with the diagram of Galileo's idea.

One last picture, perhaps the simplest of all, hints at the relation to multiplication of numbers:



Why does $3 \times 2 = 6$?

I hope these pictures seem suggestive to you. Our goal is to learn to use them as precise instruments of understanding and reasoning, not merely as intuitive guides.

Exercise 1:
Find other examples of combining two objects to get a third. Which of them seem to fit our pattern? That is, for which of them does the third object seem to have 'maps' to the two you began with? It may be helpful to start by thinking of real-life problems for which multiplication of numbers is needed to calculate the solution, but not all examples are related to multiplication of numbers.

Exercise 2:
The part of Galileo's work which we discussed is really concerned with only a small portion of space, say the immediate neighbourhood of the tower of Pisa. Since the ground might be uneven, what could be meant by saying that two points are at the same level? Try to describe an experiment for deciding whether two nearby points are at the same level, without using 'height' (distance from an imaginary plane of reference.) Try to use the most elementary tools possible.

PART I

The category of sets

A *map* of sets is a process for getting from one set to another. We investigate the *composition* of maps (following one process by a second process), and find that the algebra of composition of maps resembles the algebra of multiplication of numbers, but its interpretation is much richer.

ARTICLE I

Sets, maps, composition

A first example of a category

Before giving a precise definition of 'category', we should become familiar with one example, the **category of finite sets and maps**.

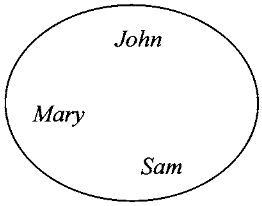
An [object] in this category is a finite *set* or *collection*. Here are some examples:

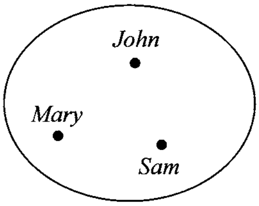
(the set of all students in the class) is one object,
(the set of all desks in the classroom) is another,
(the set of all the twenty-six letters in our alphabet) is another.

You are probably familiar with some notations for finite sets:

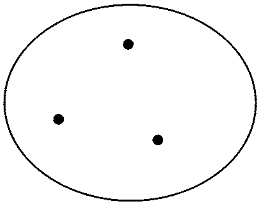
$$\{John,\ Mary,\ Sam\}$$

is a name for the set whose three elements are, of course, John, Mary, and Sam. (You know some infinite sets also, e.g. the set of all natural numbers: $\{0, 1, 2, 3, \ldots\}$.) Usually, since the order in which the elements are listed is irrelevant, it is more helpful to picture them as scattered about:





where a dot represents each element, and we are then free to leave off the labels when for one reason or another they are temporarily irrelevant to the discussion, and picture this set as:



Such a picture, labeled or not, is called an *internal diagram* of the set.