Chapter 16

Kinship and Mathematical Categories

F. William Lawvere

Those concepts that are historically stable tend to be those that in some way reflect reality, and in turn tend to be those that are teachable. "Mathematical" should mean in particular "teachable," and the modern mathematical theory of flexible categories can indeed be used to sharpen the teachability of basic concepts. A concept that has enjoyed some historical stability for 40,000 years is the one involving that structure of a society that arises from the biological process of reproduction and its reflection in the collective consciousness as ideas of genealogy and kinship. Sophisticated methods for teaching this concept were devised long ago, making possible regulation of the process itself. A more accurate model of kinship than heretofore possible can be sharpened with the help of the modern theory of flexible mathematical categories; at the same time, established cultural acquaintance with the concept helps to illuminate several aspects of the general theory, which I will therefore try to explain concurrently.

Abstracting the genealogical aspect of a given society yields a mathematical structure within which aunts, cousins, and so on can be precisely defined. Other objects, in the same category of such structures, that seem very different from actual societies, are nonetheless shown to be important tools in a society's conceptualizing about itself, so that for example gender and moiety become labeling morphisms within that category. Topological operations, such as contracting a connected subspace to a point, are shown to permit rationally neglecting the remote past. But such operations also lead to the qualitative transformation of a topos of pure particular Becoming into a topos of pure general Being; the latter two kinds of mathematical toposes are distinguished from each other by precise conditions. The mythology of a primal couple is thus shown to be a naturally arising didactic tool. The logic of genealogy is not at all 2-

valued or Boolean, because the truth-value space naturally associated with the ancestor concept has a rich lattice structure.

As John Macnamara emphasized, an explicit mathematical framework is required for progress in the science of cognition, much as multidimensional differential calculus has been required for 300 years as a framework for progress in such sciences as thermomechanics and electromagnetics. I hope that the examples considered here will contribute to constructing the sort of general framework John had in mind.

Although a framework for thinking about thinking would be called "logic" by some ancient definitions, in this century logic has come to have a much more restrictive connotation that emphasizes statements as such, rather than the objects to which the statements refer; this restrictive connotation treats systems of statements almost exclusively in terms of presentations (via primitives and axioms) of such systems, thus obscuring the objective aspects of abstract generals that are invariant under change of presentation. About 100 years ago the necessary objective support for this subjective logic began to be relegated to a rigidified set theory that shares with mereology the false presupposition that given any two sets, it is meaningful to ask whether one is included in the other or not. Although the combination of narrow logic and rigidified set theory is often said to be the framework for a "foundation of mathematics," it has in fact never served as a foundation in practice; for example, the inclusion question is posed in practice only between *subsets* of a given set, such as the set of real numbers or the set of real functions on a given domain. There are many different explicit transformations between different domains, but the presumption that there is a preferred one leads to fruitless complications.

The development of multidimensional differential calculus led in particular to functional analysis, algebraic geometry, and algebraic topology, that is, to subjects whose interrelationships and internal qualitative leaps are difficult to account for by the narrow and rigidified "foundations." This forced the development of a newer, more adequate framework (which of course takes full account of the positive results of previous attempts) 50 years ago, and 20 years later the resulting notions of category, functor, natural transformation, adjoint functor, and so on had become the standard explicit framework for algebraic topology and algebraic geometry, a framework that is even indispensable for the communication of many concepts. Then, inspired by developments in mechanics and algebraic geometry, it was shown how a broader logic and a less rigid notion of set are naturally incorporated in the categorical framework.

Toposes are categories of sets that have specified internal cohesion and variation and whose transformations are continuous or equivariant. The infinite variety of toposes now studied arises not in order to justify some constructivist or other philosophical prejudice, but in order to make usefully explicit the modes of cohesion and variation that are operative in multidimensional differential calculus. The internal logic of toposes turned out to be that which had been presented earlier by Heyting. Heyting logic (and co-Heyting logic) describes inclusions and transformations of subsets of such sets in a way that takes account of the internal cohesion and variation, refining the fragmented and static picture that the still earlier approximation by Boolean logic provides.

I will explore here two examples showing how the categorical insight can be used to make more explicitly calculable, not only the more overtly spatial and quantitative notions, but also other recurring general concepts that human consciousness has perfected over millennia for aiding in the accurate reflection of the world and in the planning of action. The two examples are kinship, in particular, and the becoming of parts versus the becoming of wholes, in general. Since the narrowing of Logic 100 years ago, it has been customary to describe kinships in terms of abstract "relations"; however, if we make the theory slightly less abstract, the category of examples gains considerably in concreteness and in power to represent concepts that naturally arise. Becoming has long been studied by resolving it into two aspects, time and states, with a map of time into the states; but when we consider the becoming of a whole "body" together with that of its parts (e.g., the development of collective consciousness or the motion of the solar system), there are several "conflicting" aspects that we need to manage: the collective state "is" nothing but the ensemble of the states of the parts, yet the constitutive law determining what the state (even of a part) "becomes" may depend on the state of all; the time of the whole may be taken as a prescribed synchronization of the times of each part, yet the news of the collective becoming may reach different parts with differing delays. A general philosophical idea is that the cohesiveness of being is both the basis in which these conflicts take place and also partly the result of the becoming (Hegel: "Wesen ist gewesen"—that is, the essence of what there is now is the product of the process it has gone through). I will try to show more precisely how the being of kinship in a given historical epoch relates to the becoming of reproduction and of intermarriage between clans, even when this kinship and reproduction are considered purely abstractly, neglecting their natural and societal setting.

16.1 A Concrete Category Abstracting the Notion of Reproduction

I want to consider general notions of kinship system, as suggested by known particular examples. The following fundamental notion was elaborated in collaboration with Steve Schanuel (Lawvere and Schanuel 1997).

An analysis of kinship can begin with the following biological observation:

(T_1) Each individual has exactly one mother and exactly one father, who are also individuals.

As a first approximation, we can explore the ramifications of this idea, taken in itself, as an abstract theory T_1. Though we may have thought originally of "individual" as meaning a member of a particular tribe, there is in fact in a "theory" no information at all about what individuals are; the mathematical concrete corresponding to the stated abstract theory T_1 is the category whose objects are sets, consisting of abstract elements, but structured by two specified self-mappings m and f. Such a structured set can be considered as a "society." We can immediately define such concepts as "a and b are siblings" by the equations $am = bm$ and $af = bf$, or "b's maternal grandfather is also c's paternal grandfather" by the equation $bmf = cff$. (In the Danish language 'morfar' means mother's father, 'farfar' means father's father, similarly 'farmor' and 'mormor'; and young children commonly address their grandparents using those four compound words.) There is a great variety of such systems, most of which seem at first glance to be ineligible, even mathematically, for more than this verbal relation to the kinship concepts; for example, I cannot be my own maternal grandfather. However, if we take the theory and the category seriously, we will find that some of these strange objects are actually useful for the understanding of kinship.

As is usual with the concrete realizations of any given abstract theory, these form a category in the mathematical sense that there is a notion of structure-preserving morphism $X \xrightarrow{\phi} Y$ between any two examples: in this case the requirement on ϕ is that $\phi(xm) = \phi(x)m$ and $\phi(xf) = \phi(x)f$ for all x in X. (It is convenient to write morphisms on the left, but structural maps on the right of the elements; in the two equations required, the occurrences of m and f on the right sides of the equality refer to the structure in the codomain Y of the morphism.) There are typically many morphisms from given domain X to given codomain Y. If $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$

are morphisms, there is a well-determined composite morphism $X \xrightarrow{\psi\phi} Z$, and for each object X, there is always the identity morphism from X to itself. Two objects X_1 and X_2 are isomorphic in their category if there exists a pair of morphisms $X_1 \to X_2 \to X_1$ with both composites equal to the respective identity morphisms. Most morphisms are not isomorphisms, and they indeed do not even satisfy the requirement of "injectivity," which just means that the cancellation law, $\phi x_1 = \phi x_2$ implies $x_1 = x_2$, holds for ϕ. If a morphism ξ is in fact injective, it is also called "a subobject of its codomain, which taken in itself is the domain." (Contrary to some rigidified versions of set theory, a given object may occur as the domain of many different subobjects of another given object. A subobject may be thought of as a specified way of "including" its domain into its codomain.) An element y is said to "belong to" a subobject ξ (notation: $y \varepsilon \xi$) if there is x for which $y = \xi x$; this x will be unambiguous because of the injectivity requirement on ξ. Here by an element of X is often meant any morphism $T \to X$; such elements are also referred to as figures in X of shape T, but often we also restrict the word 'element' to mean 'figure of a sufficient few preferred shapes'. Sometimes we have between two subobjects an actual inclusion map $\xi_1 \subseteq \xi_2$ *as* subsets of Y (i.e., $X_1 \xrightarrow{\alpha} X_2$ for which $\xi_1 = \xi_2\alpha$). It is these inclusions that the propositional logic of Y is about.

16.2 Genealogical Truth, Seen Topos-Theoretically

The category sketched above, that is, the mathematically concrete correspondent of the abstract theory T_1 of m and f, is in fact a "topos." That means, among other properties, that there is a "truth-value" object Ω that "classifies," via characteristic morphisms, all the subobjects of any given object. In the simpler topos of abstract (unstructured) sets, the set 2, whose elements are "true" and "false," plays that role; for if $X \xrightarrow{\xi} Y$ is any injective mapping of abstract sets, there is a unique mapping $Y \xrightarrow{\phi} 2$ such that for any element y of Y, $\phi y =$ true if and only if y belongs to the subset ξ; and if two maps ϕ_1, ϕ_2 from Y to 2 satisfy ϕ_1 entails ϕ_2 in the Boolean algebra of 2, then the corresponding subsets enjoy a unique inclusion map $\xi_1 \subseteq \xi_2$ as subsets of Y (i.e., $X_1 \xrightarrow{\alpha} X_2$ for which $\xi_1 = \xi_2\alpha$). However, in our topos, 2 must be replaced by a bigger object of truth-values in order to have those features for all systems Y and for all subsystems X, ξ, as explained below.

In our topos of all T_1-societies, there is a particular object I, which as an abstract set consists of all finite strings of the two symbols m and f, with the obvious right-action that increases length by 1, giving its cohesiveness and variation; in particular, the empty string 1_I is the generic individual, all of whose ancestors are *distinct*. A concrete individual in a society X is any morphism $I \to X$; in particular, the infinitely many endomorphisms $I \xrightarrow{w} I$ are easily seen to be in one-to-one correspondence with the strings in I. Composing with the two endomorphisms m and f, which correspond to the two strings of length 1, completely describes, in the context of the whole category, everything we need to know about the particular internal structure of X: the actions xm, xf are realized as special cases of the composition of morphisms. The condition that every morphism $X \xrightarrow{\phi} Y$ in the topos must satisfy is then seen to be just a special case of the associative law of composition, namely, the special case in which the first of the three morphisms being composed is an endomorphism of I. The facts stated in this paragraph constitute the special case of the Cayley-Yoneda lemma, applied to the small category W that has one object, whose endomorphisms are the strings in two letters, composed by juxtaposition; a category with one object only is often referred to as a *monoid*.

With loss in precision we can say that x' is an ancestor of x in case there exists some endomorphism w of I for which $x' = xw$. In the particular society X, the w may not be unique; for example, $xmff = xfmf$ might hold. But the precision is retained in the category W/X, discretely fibered over W, which is essentially the usual genealogical diagram of X. The correlation of the reproductive process in X with past solar time could be given by an additional age functor from this fibered category to a fixed ordered set; a construction of Grothendieck (1983) would permit internalizing such a chronology too in T_1's topos.

Now we can explain the truth-value system. The subsets of I in the sense of our topos are essentially just sets A of strings that, however, are not arbitrary but subject to the two conditions that if $I \xrightarrow{w} A$ is a member of A, then also wm and wf must be members of A. Thus, A can be thought of as all my ancestors before a certain stage, the antiquity of that stage depending however on the branch of the family. These As constitute the truth-values! To justify that claim, we must first define the action of m and f on As; this is what is often thought of as "division":

$$A : m = \{w \in W : mw \in A\},$$

$$A : f = \{w \in W : fw \in A\}.$$

It is easily verified that these two are again subobjects of I if A is. Note that $A : f$ consists of all my father's ancestors who were in A. The truth-value true is $A = W$, the whole set of strings, and union and intersection $A \cup B$, $A \cap B$ are the basic propositional operations *or*, *and* on truth-values. The operation *not* A of negation does not satisfy all the Boolean laws, since it must again yield a subobject of I:

$$not\ A = \{w \in W : \text{for all } v \in W, wv \text{ does not belong to } A\}$$

is the set of all my ancestors none of whose ancestors were in A, so (A *or not* A) is usually much smaller than true. Why does the object Ω thus defined serve as the unique notion of truth-value set for the whole topos? Consider any subset $X \xrightarrow{\xi} Y$ in the sense of the topos; then there is a unique morphism $Y \xrightarrow{\phi} \Omega$ such that for all individuals $I \xrightarrow{y} Y$ in Y,

$$y \in \xi \text{ if and only if } \phi(y) = \text{true}.$$

The value of ϕ on any element y is forced to be

$$\phi(y) = \{w \in W : yw \in \xi\}.$$

This is the measured answer to the question "Precisely how Irish is y?"

16.3 How to Rationally Neglect the Remote Past

Every X is a disjoint union of minimal components that have no mutual interaction; if we parameterize this set of components by a set $\Pi_0(X)$ and give the latter the trivial (identity) action of m, f, then there is an obvious morphism $X \to \Pi_0(X)$. If $\Pi_0(X) = 1$, a one-point set, we say X is connected; that does not necessarily mean that any two individuals in X have some common ancestor, because more generally connecting can be verified through cousins of cousins of cousins, and so on.

An important construction borrowed from algebraic topology is the following:

Given a subobject $A \hookrightarrow X$, we can form the "pushout" $X\, mod\, A$ that fits into a commutative rectangular diagram involving a collapsing morphism $X \to X\, mod\, A$ and an inclusion $\Pi_0(A) \hookrightarrow X\, mod\, A$ as well as the canonical $A \to \Pi_0(A)$ and satisfies the universal property that, if we are given any morphism $X \xrightarrow{\psi} Y$ whose restriction to A depends only on $\Pi_0(A)$, then there is a unique morphism $X\, mod\, A \xrightarrow{\psi_0} Y$ for which ψ is the composite, ψ_0 following $X \to X\, mod\, A$, and which also agrees on $\Pi_0(A)$ with that restriction.

This $X\, mod\, A$ is a very natural thing to consider because practical genealogical calculations cannot cope with an infinite past. For example, the Habsburg X are mainly active over the past 1,000 years and the Orsini over the past 2,000 years. But this collapsing of the remote ancestors A to $\Pi_0(A)$ introduces an interesting idealization; for simplicity we consider the case where A is connected, $\Pi_0 A = 1$. Then the construction has given rise to a morphism $1 \to X\, mod\, A$ that is the residuum of A. But what is a "point" (= figure of shape 1) in a topos like ours? Since m and f act as identity in 1, a point (e.g., the point of $X\, mod\, A$ that has arisen) is a special sort of individual x that is its own mother and father:

$$xm = x = xf.$$

This idea of a superindividual is of course merely a convenience in genealogical bookkeeping.

16.4 Deepening the Theory to Make Gender Explicit

A reasonable gender structure is not definable in the abstract theory T_1 above, since there is no requirement that the values of the operator m be disjoint from the values of the operator f and also no way of determining the gender of infertile individuals. So as usual we refine our theory to a richer one:

(T_2) In addition to T_1, every individual has a definite gender, every individual's mother is female, and every individual's father is male.

The corresponding category of all concrete applications of T_2 will also be a topos and in fact have a direct relation to the topos of T_1. Namely, in the first topos there is a particular object G with only two individuals called *male* and *female* and where the two structural operations act as the constant maps. Our refined topos has objects that may be described as pairs X, γ where $X \xrightarrow{\gamma} G$ is a morphism in the first topos and has as morphisms all triangular diagrams $[X \xrightarrow{\gamma} G, X \xrightarrow{\phi} Y, Y \xrightarrow{\delta} G]$ in the first topos for which $\delta\phi = \gamma$; the codomain of this triangle is defined to be the pair Y, δ. The conditions

$$\gamma(xm) = (\gamma x)m$$

and

$$\gamma(xf) = (\gamma x)f$$

that morphisms must satisfy state in this case that γ is a compatible gender-labeling, and the further condition on a triangular diagram means that morphisms in our refined topos moreover preserve gender. That applies in particular to the individuals in X that happen to be (in our purely reproductive sense) spinsters or bachelors. This topos has two generic individuals, because labeling maps $I \to G$ can map 1_I either to m or to f. The terminal object of this new topos is actually 1_G so that a point in the new sense is really the "locus of a point moving virtually between m and f"; more exactly, a morphism $1_G \to Y$ in this topos amounts to an Eve/Adam pair for which

$$ef = af = a \quad \& \quad em = am = e.$$

If we construct the pushout along $A \to \Pi_0 A$ of a subobject $A \hookrightarrow X$ in the sense of this topos, we get in particular $\Pi_0 A \hookrightarrow X/A$, which specifies a different Eve/Adam pair for each of the mutually oblivious subsocieties among the specified ancestors.

16.5 Moieties

A still further refinement makes precise the idea of a society equipped with a strategy for incest avoidance. The simplest idea T_3 of such is that of (matrilineal) moiety labeling, which can be explained in terms of another two-individual application C of T_1. The two "individuals" are called Bear and Wolf in Lawvere and Schanuel 1997. The idea is that the moiety of any individual x is the same as that of x's mother and distinct from that of x's father. Thus, on the two individuals in C, m acts as the identity, whereas $f^2 = \text{id}$ but $f \neq \text{id}$. Any T_1-respecting morphism $X \to C$ is a labeling with the expected properties, and commuting triangles over C constitute the morphisms of a further topos that should be investigated. However, I will denote by T_3 the richer abstract theory whose applications are all the T_1-applications X equipped with a labeling morphism $X \to G \times C$. Thus, in T_3's topos there are four generic individuals I (one for each pair in $G \times C$), morphisms from which are concrete individuals of either gender and of either Bear or Wolf moiety.

16.6 Congealing Becoming into Being

According to the criteria proposed in Lawvere 1991, the three toposes thus far introduced represent particular forms of pure becoming, because they satisfy the condition that every object X receives a surjective morphism from another one E that has the special property

$$z_1 w = z_2 w \quad \text{implies } z_1 = z_2$$

for any given w in W and any two individuals z_1, z_2 in E. This "separable covering" E of X is achieved through refining each individual (if necessary) into several individuals with formally distinct genealogies (e.g., $xmf \neq yf$ in E even if $y = xm$ in X). However, by restricting to certain subcategories consisting only of objects that contain something roughly like the Eve/Adam pair, we obtain toposes that have instead the qualitative character of pure being; for example, the Habsburg era had a certain quality of being (and was the basis for a lot of motion).

More precisely, let us consider various theories corresponding to monoid homomorphisms

$$W \xrightarrow{p} M$$

that are epimorphic. Epimorphic homomorphisms may be surjective (induced by a congruence on W), or may adjoin at most new operator symbols that are two-sided inverses of operator symbols coming from W, or may involve a combination of these two kinds of "simplifications" of W. Such a homomorphism gives rise to a well-defined subcategory of T_1's topos, namely, the one consisting of all those systems X satisfying the congruence or invertibility conditions that become true in M. (For analogous subcategories of the other two toposes, we would need to consider epimorphic functors

$$W/G \xrightarrow{p} D$$

and

$$W/G \times C \xrightarrow{p} E$$

to small categories D or E of two (respectively four) objects.) The resulting subcategories, although toposes, are not subtoposes but "quotient toposes." Here a quotient morphism of toposes, compressing a bigger situation into a smaller situation, involves a full and faithful inverse functor p^* (the obvious inclusion in our example) and two forward functors $p_!$ and p_* that are respectively left and right adjoint to p^* in the sense that there are natural one-to-one correspondences

$$\frac{p_! X \to Y}{X \to p^* Y} \qquad \frac{Z \to p_* X}{p^* Z \to X}$$

between morphisms, for all sets X defined over the big situation and all Y, Z defined over the smaller one. Note that the notion of morphism is the same in both situations; that is what it means to be full and faithful.

These adjointnesses are the refined objective version of relations whose subjective reflections are the rules of inference for existential quantification $p_!$, resp. universal quantification p_* of predicates relative to a substitution operation p^* (indeed, p^* preserves coproducts and products of objects, as is subjectively reflected in the fact that a substitution preserves disjunction and conjunction of predicates; it may fail to preserve the analogous negations however, for example in the case of continuous predicates in topology). Specifically, p_*X extracts those individuals from X whose genealogy happens to conform to the requirements specified in M, whereas $p_!X$ applies to all individuals in X, the minimal forcing required to merge them into a new object that conforms to M.

These facts will be used below in investigating elementary examples of quotient toposes, in particular, some of those that seem to congeal becoming into being.

To distinguish a topos of general pure being, I proposed in Lawvere 1986 and 1991 the two criteria that $\Pi_0(X \times Y) = \Pi_0(X) \times \Pi_0(Y)$ and that every object is the domain of a subobject of some connected object. The first of these means that any pair consisting of a component of X and a component of Y comes from a unique single component of the Cartesian product $X \times Y$; in particular, the product of connected objects should be again connected (since $1 \times 1 = 1$), which is rarely true in a topos of particular becoming. That condition will be important for the qualitative, homotopical classification of the objects, but I will not discuss it further here.

The second criterion mentioned above, that every object can be embedded as a subobject of some connected object, has the flavor "We are all related," but is much stronger than merely requiring that a particular object be connected; for example, the disjoint sum of two connected objects, which is always disconnected, should be embeddable in a connected object.

It follows from a remark of Grothendieck (1983) that both proposed criteria will be satisfied by a topos of M-actions if the generic individual I ($= M$ acting on itself) has at least two distinct points $1 \rightrightarrows I$. The distinctness must be in a strong sense that will be automatic in our example where 1 has only two subobjects; it is also crucial that the generic individual I itself be connected. Part of the reason why this special condition of Grothendieck implies the two general criteria can be understood in the following terms: it is not that becoming is absent in a topos of general being; rather, it is expressed in a different way. An object X may be a space of locations or states in the tradition of Aristotle; that is, X is an arena in which becoming can take place. In particular, some such spaces T can measure time intervals and hence a particular motion in X during T may give rise to a morphism $T \xrightarrow{\mu} X$ describing the result of the becoming process; that is, if T is connected, if we can distinguish two instants $1 \rightrightarrows T$ called t_0, t_1 and if $\mu t_k = x_k$ for $k = 0, 1$, then we can say that x_0 became x_1 during the process μ; this is the common practice in mathematical engineering (at least for certain categories within which it has, in effect, become customary to work in the past 300 years). But then if we consider t_1 as a subobject of T, its characteristic map from T to the truth-value space will be a process along which false becomes true; this in turn implies that the truth-value space is itself connected since T is, and because of the propositional structure of truth-values, it follows that indeed the hyperspace, whose points parameterize the subspaces of X, is connected too. On the other hand, in any topos any object X is embedded as a subobject of its hyperspace via the "singleton" map. Taking $T = I$, it follows that in the cases under discussion every object can be embedded in an object that is not only connected, but moreover has no holes or other homotopical irregularities. Note that now each individual $I \xrightarrow{x} X$ involves also a process in his or her society. There are many such situations, but what can such a process mean?

As a simple example, suppose that we want a category of models of societies in which genealogical records go back to grandparents. Thus, we consider the homomorphism $W \xrightarrow{p} M$ onto a seven-element monoid in which

$$m^3 = mfm = m^2$$

and

$$mf^2 = m^2 f = mf$$

hold, together with similar equations with f and m interchanged. Drawing the obvious genealogical diagram, we see that the generic individual I in the topos of M actions has parents that represent the first intermarriage between two distinct lines, and that I's grandparents are the "Eves and Adams" of those two distinct lines. Indeed, this object I permits a unique gender structuring and so we may pass instead into the topos whose theory is M/G, finding there that each of the two generic individuals I_G has two distinct points $1_G \rightrightarrows I_G$. The "process" involved in an individual boy $I_G \xrightarrow{x} X$ in a society X in this topos is that whereby his father's line united with his mother's.

16.7 Finer Analysis of the Coarser Theories

Given any object L in a topos, one can form a new topos of objects further structured by a given labeling morphism to L. However, it is worthy of note that in our examples, one $L = G$ (leading to T_2) and the other one $L = G \times C$ (leading to T_3), these labeling objects belong to a much restricted subcategory (quotient topos) within the topos of all T_1-objects. Although the T_1 topos, consisting of all right actions of the free monoid W on two symbols, is too vast to hope for a complete survey of its objects (as is borne out by theorems of Vera Trnková stating that any category can be embedded as a full subcategory of it; see Pultr and Trnková 1980), by contrast these smaller toposes are more tractable.

Consider first the case G of gender in itself. It belongs to the subcategory corresponding to $W \to W_3$, the three-element monoid in which $xf = f$, $xm = m$ for all three $x = 1, f, m$. If we restrict attention to the right actions of W_3 only, we see that the generic individual I in the sense of that topos has only one nontrivial subobject, which is G itself. Therefore, the truth-value space Ω_3 for that topos (in contrast to the infinite Ω for W) has only three elements also. The most general "society" X is a sum of a number of noninteracting nuclear families, some with two parents and some with a single parent, and a society is determined up to isomorphism by the double coefficient array that counts families of all possible sizes of these two kinds. Cartesian products are easily computed.

Second, consider the simple moiety-labeler C in itself; it belongs to the quotient topos determined by $W \to W_2$, the two-element group generated by f with $f^2 = 1$, where we moreover interpret m as 1. Here C is itself the generic individual and has no nontrivial subobjects, so that Ω_2 is the two-element Boolean algebra. All right W_2-actions are of the form $X = a + bC$, where a is the number of individuals fixed, and b half the number moved, by f. These objects are multiplied by the rule $C^2 = 2C$.

Third, we can include both G and C in a single subcategory as follows: the infinite free monoid W maps surjectively to each of W_3 and W_2; therefore, it maps to the six-element product monoid $W_3 \times W_2$, but not surjectively since the image W_5 is a five-element submonoid (the missed element is the only nonidentity invertible element in the product monoid). The category of right W_5-sets can be probed with the help of its generic individual I, which again (surprisingly) has only three subobjects so that Ω_5 has again three truth-value-individuals. In I (i.e., W_5) there are besides me my four distinct grandparents, the T_1-structure coming from the fact that my parents are identical with my maternal grandparents in this theory. The unique nontrivial subobject of I is precisely $G \times C$, the label object for our central noncollapsed theory T_3; it is generated by any single one of its four elements. I leave it to the reader to determine a structural description of all right W_5-actions, with or without $G \times C$ labelings.

16.8 Possible Further Elaborations

To the T_2 topos (whose abstract general is the two-object category W/G) we can apply separately our concrete construction idealizing remote ancestors and also our refinement of the abstract general by moiety-labeling. However, it does not seem consistent to apply both of these simultaneously, owing to the simple group-theoretic nature of our C. A reasonable conjecture is that tribal elders in some part of the world have devised a more subtle abstract general C' that relaxes the incest avoidance for the idealized remote ancestors, thus permitting a smoothly functioning genealogical system enjoying both kinds of advantages.

Some more general examples of simplifying abstractions include that based on a homomorphism $W \xrightarrow{p} M_n$ that makes every string of length greater than n (where, e.g., $n = 17$) congruent to a well-defined associated string of length n. An interesting construction to consider is the application of the pushout construction to an arbitrary T_1-object X, not with respect to some arbitrarily chosen remoteness of ancestry A, but with respect to the canonical morphism $p^*p_*X \to X$.

The need for Eve and Adam arises from the central role of self-maps as structure in the above theories and might be alleviated by considering instead a two-object category as the fundamental abstract general, although possibly at the expense of too great a multiplicity of interpretations. This category simply consists of three parallel arrows, as described in Lawvere 1989. A concrete application of this theory T_0 involves two sets (rather than one) and three internal structural maps

$$X_{now} \rightarrow X_{up\ to\ now}$$

called m, f, and s, where s specifies, for each contemporary individual, the place of his or her "self" in genealogical history. There is a natural notion of morphism between two such objects (involving two set-maps subject to three equations) yielding again a topos, whose truth-value object is finite and illuminating to work out. T_0's topos has T_1's topos as a quotient, because forcing the structural map s to become invertible yields a small category equivalent to the monoid W. If s is not inverted, iteration of m and f is a more complicated partial affair: in case xm has the property that there is a contemporary mother y for which $ys = xm$, then yf is a maternal grandfather of x; but there may not be such a y, and on the other hand, mathematical experience counsels against excluding in general the possibility of several such y for a given x. If we do invert s, we make explicit that which, in a larger sense, is the theme of this book, the continuing role of each past individual in our cognition at present.

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