SOME THOUGHTS ON THE FUTURE OF CATEGORY THEORY

F. William Lawvere
Department of Mathematics, S.U.N.Y. at Buffalo
Buffalo, NY 142140

The Como meeting was something of a milestone, coming as it did just twenty-five years after the first international meeting on category theory held at La Jolla, California in 1965. The work of Kan, Grothendieck, and others had greatly intensified the elaboration and application of the subject in the ten years prior to La Jolla, and enormous development has continued uninterruptedly since. I have been asked, as a participant at both meetings, to speculate on how at least some of the threads of the subject might develop in the immediate future. The threads I have selected now were only dimly visible then, for when J. L. Verdier described topos theory on the beach at La Jolla, most of us were slow to grasp its significance.

The crystallized philosophical discoveries which still propel our subject include the idea that a category of objects of thought is not specified until one has specified the category of maps which transform these objects into one another and by means of which they can be compared and distinguished. Thus, for applications of mathematics, to objectify is to mapify. Quite non-trivial in fact is also the idea that there must be definite domains and definite codomains and that there must be identity maps; even today there are many who think one could usefully "generalize" by omitting those requirements, sometimes on grounds of dislike for the "stasis" they think they imply. However, in modern Greek "stasis" means "bus-stop"; how useless an intricate network of speeding buses would be without them, and how disembodied would be processes without states. In fact category theory is the first to capture in reproducible form an incessant contradiction in mathematical practice: we must, more than in any other science, hold a given object quite precisely in order to construct, calculate, and deduce; yet we must also constantly transform it into other objects. These precepts, together with the powerful guide to look for and use adjoints in all categories large and small because they are the form of most constructions and deductions and many calculations and estimates, have guided us in our work in all the varied fields of mathematics. Most of us have struggled to explicitly introduce these principles also into our teaching, and those who have persisted find that this explicit use of the unity and cohesiveness of mathematics sparks the many particular processes whereby ignorance becomes knowledge, in learning just as it does in investigating. The need to teach, to explain and to respond to students' probing is often the genesis of problems taken up in "pure" research.

Though much remains to be done, it seems to some that we (that is, the community of category theorists with our ties to all the fields of pure and applied mathematics) have reached a unique position with regard to philosophy. I concentrate here on an outline of what is intended as a positive mathematical program. The history of possible philosophical objections to it will be treated elsewhere. Suffice it to suggest that Möbius, Hamilton, Grassman, Maxwell, etc. would not be among the naysayers. At least we can hope that sober application, of category theory to the ancient philosophical categories, will not only clarify both but also renew respect for serious thought, through solid examples approaching adequacy to their concept.

This attempt, by an admirer of rational mechanics, to include objective logic among the tools for arriving at a more accurate conception of space, will, I hope, not be dismissed by confusing it with objective idealism. The general science of the development of scientific ideas has a big overlap with category theory. That general science does not claim that scientific ideas are self-generating nor does it depend on faith for the acceptance of its own conclusions, as idealism would.

In the first section I start from the opposition between connected and separable objects to propose the tentative clarification, by a certain disjoint pair of classes of categories, of the conceptions of Being and Becoming respectively; how the one class arises from the other is the content of some resulting mathematical conjectures. In the second section, a specific mathematical formulation of the principle "unity-and-identity-of-opposites" is described in hopes of clarifying dimensionality in general and infinitesimals in particular, with again some mathematical conjectures aiming at further clarification. In the third section it is urged that certain pathologies "commonplace" since 1861/1890 need not be included in a more accurate conception of space and that both more physically-realistic models of computers as well as a more "objective" approach to Diophantine problems are already emerging from certain fascinating calculations.

I. In the remarkable paper [QDC] (Quotients of Decidable Objects, Cambridge) a certain epsilon difference between classes of toposes is mentioned. This epsilon is in a way the victory of geometry over narrow logicism and is what [QDB] (Qualitative Distinctions, Boulder) is groping to clarify. There were actually two kinds of mathematical examples which around 1960 forced the qualitative generalization of the previous notion of sheaf: for a particular algebraic space, the replacement of mere open subspaces by objects unramified over it, and for the category of all analytic spaces, the enlargement to a much nicer *category* within which those *particular* spaces which happen to be "nicer" according to some previously-achieved definition could be styled "representable". This raised the question: given a space in a topos of the second kind, what is the reasonable topos of pseudo-classical sheaves on it? Taking only a few of the simplest ideas from those circulated by the dozen or so who have strenuously worked on this problem, one arrives at the suggestion to study QD *sub*toposes of the topos of all spaces over a given space. Let's recall what QD means.

While the term "decidable" has subjective connotations which are a powerful guide to certain investigations, and "separable" is well-established in commutative algebra, in geometry "unramified" has an objective history; here I may use "SUD-object" to remind myself of the essential identity of all three. For brevity I'll use the term "neat" for objects which are *both* SUD and connected. There is a reflection from any locally connected topos to a topos in which every object is a colimit of neat objects; on the other hand, there is no coreflection (like Booleanization). In the cases where the reflection map is local, we have a start on the investigation of QD subtoposes. Note that "locally connected" and "sum" are relative to a base topos, which itself is quite special but (as in the Galois base of algebraic geometry) is sometimes much better *not* the topos of abstract sets; it would be fortunate if the hypothesis that the base itself be QD turned out to be sufficient . As usual, if I say "set", we should imagine an object of the base topos.

To clarify the above considerations, generalize to distributive categories and seek philosophical guidance. Even though the determination of which maps are epimorphisms is the more profound question studied with Grothendieck topologies, it takes place within a topos of the following kind. Call a small category C "extensive" if it has finite coproducts which yield an equivalence C/A+B = C/A x C/B and C/0 = 1 (this seems a minimum requirement on an op-fibration to conform with the notion of "family" and with Grassmann's "combinatorics of continuous magnitudes"); for example, "the" homotopy category or the category of spaces of dimension at most 4.

Then the topos G(C) of all those presheaves X on C for which merely X(A+B) = X(A) x X(B) & X(0) = 1 contains any conceivable theory of intensive quantities, either cohomological or function-theoretic, as an (algebra) *object.*

Further subtoposes of G(C) and G(C)/X based on "flatness" or the need to classify structures satisfying existential axioms are obviously of great importance, but passed over here to get to the main point. Those extensive categories which also have finite limits are called distributive, as discussed in the paper on their Burnside rigs in this volume [ECD].

A general category of Being, particular categories of Becoming: this is a suggested philosophical guide for sorting the two original kinds of toposes and what they have become. The unity and cohesiveness of Being provides the basis for Becoming, and the historicity and controlled variability of Becoming produces new Being from old. The unity and cohesiveness of space suggests the following condition on a category: "Every object can be included in a connected object". This axiom is not true in many toposes, for example, in the Weil (or infinitesimal) topos. If the category is not a topos, the axiom should perhaps be strengthened to say that every object is an equalizer of maps between connected objects; if the category is a topos, the axiom can be sufficiently checked on the one object 1 + 1, but implies that any object is functorially included in a contractible object (i.e. C for which all C^X are connected). This axiom for a category of Being will be paired with another one.

Here is a dialogue which suggests how the unity-and-cohesiveness axiom may be used: Suppose you claim that the surface of the earth and the point called the sky have "nothing to do with each other", whereas I claim they must. As a first step I consider the contractible container C of E+S which (though simple) may then become the basis for a more concrete connection, such as a scheme C' for a system of airlines and airports. . . . .[By the way, the unity-and-cohesiveness axiom can sometimes be demonstrated without invocation of power sets by using properties of the four adjoint functors: components, discrete, points, codiscrete; namely, if points map surjectively to components and if each discrete space maps injectively to the codiscrete space it negates, and finally, if non-empty codiscrete spaces are connected, then we need the injectivity of pushouts of injections.]

The connected objects and the unramified objects are "orthogonal" in the sense that any two maps $C \rightrightarrows U$ are either totally equal or nowhere equal. Hence the subcategory of neat objects in a given distributive category is always a category in which every map is an epimorphism, i.e. a [QDC] site. The orthogonal axiom, that "every object X can be covered by a SUD object U", is proposed as a characterization of a category of Becoming. Why? Considering the maps between objects in the site as control processes or deformations, a figure $U \longrightarrow X$ in a sheaf X may be considered to be a state of X, and a composite $U' \xrightarrow{t} U \xrightarrow{x} X$ to be the state x' which x "becomes" under the process t. Dialogue: I think that if $x_1 t = x_2 t$ now, surely $x_1 = x_2$ originally. You say no, there are many dissipative systems X. But no, I reply, you forgot to maintain enough information about history in your definition of present state; if you correct this neglect, you will obtain an epic $hX \longrightarrow X$ where hX satisfies my original injectivity-of-becoming claim. I resolve this dialogue iff my category satisfies the above "QD" axiom (a really ineradicable dissipation would require another sort of site, possibly with "relaxation" idempotents in it.) Note that a fundamental process of analysis, where a neighborhood becomes a smaller neighborhood $U' \subset U$, inducing a section of any sheaf to become its restriction, is of this kind.

[In a distributive category, an object U is SUD iff for any two maps $A \rightrightarrows U$ the equalizer E is a coproduct summand: A = E+E'. $E' \to A$ has the property that for any map $T \longrightarrow E'$, if the composites

$$T \longrightarrow E' \longrightarrow A \rightrightarrows U$$

are equal, then T = 0. Thus the requirement that all objects in a distributive category be SUD could be extended to merely extensive categories by demanding that every pair of maps have such a pair E, E'.]

For any distributive category C and any space X in G(C), the locally distributive site of SUD objects in C/X determines a QD subtopos P(X) of G(C)/X which is an approximation to "the particular category of Becoming which X is". Of course we have plucked X from its environment, so P(X) by itself is a too-clean abstraction from which to recover X; however, the composite $P(X) \longrightarrow G(C)/X \longrightarrow G(C)$, which we may call $\mathcal{O}_X$, retains the ties: for any R in G(C), $\mathcal{O}_X{}^*(R)$ is the pseudo-classical sheaf of intensive quantities of type R. $\mathcal{O}_X$ as a classifying map shows that P(X) is canonically given the additional structure of a sheaf of C-algebras ("without idempotents"). Note that I still have not succeeded to describe this in a site-invariant manner starting from a given pair of toposes $\mathcal{X}$, $\mathcal{S}$ satisfying suitable axioms, with the nature of $\mathcal{X}$ itself determining the corresponding refined version of the fiber P(X). I hope that the above clarifies the problem and that the several efforts in this direction will combine to solve it.

The normalization $P(1) = \mathcal{S}$ and the QD reflection suggest that a suitable axiom on $\mathcal{X}$ might just be that its QD reflection map is local (which is similar to the possible dual axiom that $\neg\neg$ is essential). This strong localness tie persists when 1 is generalized to a discrete space. However, for X of higher dimension, the extra essentialness adjoint of the refined $P(X) \rightrightarrows \mathcal{X}/X$, while (remarkably) product-preserving, is not exact, and $\mathcal{X}/X \longrightarrow (\mathcal{X}/X)_{QD}$ may not be local; the image of the composite may be a significant topos. We'll return in the next section to the meaning of "higher dimension."

There are many distributive categories which satisfy both axioms: in that case every object X is the image $U \twoheadrightarrow X \hookrightarrow C$ of a map from a SUD object to a connected object. For example, consider the topos of quivers (i.e. irreflexive graphs). However, they don't satisfy the further requirement on a general category of Being that "the product of connected spaces is connected". For example, if A is the connected quiver with a single arrow, then $A^2 = A+2D$ where D is the naked-dot quiver.

The condition, that a category of Being should not only be cohesively unifying but also have its connected objects closed under finite product, justifies Hurewicz's definition $[X,Y] = \pi_0(Y^X)$ of the homotopy category, expressing a definite kind of qualitative aspect of spaces. Such a category of Being cannot be simultaneously also a pure category of Becoming. For, in that case, the neat objects would be subobjects of 1, the topos thus localic; but a product-preserving cocontinuous functor on a localic topos is always left exact, hence preserves any equalizer $2 \longrightarrow I \rightrightarrows I'$ of connected objects representing 2, so an inconsistency would be reached by taking π_0 as the functor.

II. The intuitive idea that any one-dimensional connected group must be abelian could probably be proved in any suitable topos. We know what "connected" means, but what is "one-dimensional" for an object?

It seems that a significant portion of algebraic geometry and differential geometry does not depend so much on the particular algebraic theory used to construct models for it but is of a more fundamental conceptual nature. "One-dimensional", like "connected", is actually a philosophical concept, related to the minimal Hegelian level of figures which must be considered within an arbitrary space in order to determine that space's connectedness.

By a level in a category of Being, I mean a ("downward") functor from it to a smaller category which has both left and right adjoints which are full inclusions. Such a pair of categories and triple of functors is a unity-and-identity-of-opposites (UIO) in the sense that the big category unites the two opposite subcategories which in themselves are identical with the smaller category. One can picture the big category as a (horizontal) cylinder, some objects of which lie on the identical right or left ends. The two ends are opposite not only because we picture them so, but for the intrinsic reason of adjointness; every object in the category lies on a unique horizontal thread, two objects lying on the same thread iff the downward functor assigns to them isomorphic objects in the smaller (or lower) category. All is determined by the one functor. If the big category is a topos, the right-hand end will automatically be a subtopos, but the "identical" left hand end will usually not be. To say that a particular object belongs to the level has two sharply opposed meanings: we may say that it is a sheaf for the level if it belongs to the right-hand end, but that it negates a sheaf for the level if it belongs to the left hand end. The two idempotent adjoint endofunctors of the big category obtained by composing the three are called the coskeleton (right) and skeleton (left) functors for the level; the skeleton and coskeleton of any given space (object) in Being provide a kind of interval, graspable at this level, within which the possibly more complex space being studied must lie. The basic starting example of all this is that where the downward functor is the unique one to the terminal category; then the whole big category of Being constitutes just one thread, the unique sheaf being the terminal object ("pure Being") and its negative being the initial object ("non Being"); the two opposed subcategories are singletons in this case.

Within a given category of Being, consider the partially-ordered class of all levels within it whose adjointness is enriched over a given base topos which is a category of Becoming and which itself has the structure of a level, to be thought of as the lowest non-trivial level; assume that both the initial object as well as the terminal object are sheaves for this base level; the general sheaves for this base level are commonly called "codiscrete" or "chaotic" objects within the big category of Being, and the subtopos of them may be called "pure Becoming". The negative objects for this level are commonly called "discrete" and the subcategory of them deserves to be styled "non Becoming". The objects of the base category (which is identical with the two opposite subcategories of pure Becoming and non Becoming when the inclusion functors are neglected) can just be called "sets". However, this base topos, although we have restricted it to be QD, is not necessarily the category of abstract sets; part of the philosophical content of the work of Galois is that, for the Being of algebraic geometry over a non-algebraically closed base field, a much more accurate picture is achieved if the base is taken to be a well-determined Boolean topos of more-subtly Becoming sets which is not of the purest abstract kind where the axiom of choice would hold. The base in fact seems in examples to be determined by the given category of Being itself, either as the latter's QD reflection with the extra localness condition supplying the right adjoint pure Becoming inclusion, or else (for example simplicial sets) as the double-negation sheaves with the extra essentialness condition supplying the left adjoint inclusion (in the latter case it is in Hegelian fashion always the smallest level for which both 0,1 are sheaves). Within the class of all levels over the base (of course it is a set in fact if the category of Being is a topos), the base itself is often further distinguished by having a still further left adjoint to its discrete inclusion, this extra functor therefore assigning to every space in Being its set of components. The downward (non faithful) functor itself we regard, of course, as assigning to any space its set of points.

The relation between the trivial level and the base level above it is only the first case of a possible strong relation between two levels which (hoping not to do too great an injustice to Hegel) I will call Aufhebung relative to the given category of Being: this is the relation between a lower level and a higher level whereby the first level is not only included (on the left and equivalently on the right, or simply that the longer downward functor factors across the shorter one) in the higher, but moreover that the longer left adjoint inclusion factors across the shorter right adjoint inclusion; equivalently, the higher coskeleton functor fixes both the skeleta and the coskeleta in the sense of the lower level. A very simple picture (not involving toposes) involves taking the basic downward functor to be any given map from a seven-element totally ordered set onto a three-element one, which just amounts to a partition of the big set into three non-empty closed intervals; an arbitrary intermediate level is simply a finer partition of each of these coarse intervals into finer subintervals, but an intermediate level is an Aufhebung of the lower one iff the following more stringent condition is satisfied: among the subintervals within each coarse interval, the left-most one is a singleton. In this simple example, as in some but not all examples involving toposes, every level has a smallest aufgehobenen level over it, which could reasonably be called "the" Aufhebung of it.

Unity-and-identity-of-opposites, the Aufhebung relation between two such within a given unity: this is a second proposed philosophical guide. It is not limited to distributive categories, nor is the dual case of an inclusion which has both left and right adjoint retractions without interest; that dual relation holds for example between graded modules and chain complexes, and it is the image of the canonical map between the opposites which defines the homology functor.

Having described a basic framework, we can now return to the question of the intrinsic meaning of "one-dimensionality" of an object within such a framework. The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level. Thus a zero-dimensional space is just a discrete one (there are several answers, not gone into here, to the objection which general topologists may raise to that) and dimension one is the Aufhebung of dimension zero. Because of the special feature of dimension zero of having a components functor to it (usually there is no analogue of that functor in higher dimensions), the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of of components of the skeleton at that dimension of the space, or more pictorially: if two points of any space can be connected by anything, then they can be connected by a curve. Here of course by "curve" we mean any figure in (i.e. map to) the given space whose domain is one-dimensional.

Continuing, two-dimensional spaces should be those negating a subtopos which itself contains both the one-dimensional spaces and the identical-but-opposite sheaves which the one-dimensional spaces negate.

If by "function" we mean a map with one-dimensional codomain, then any function naturally defined along each of the surfaces in an arbitrary space uniquely extends to a smooth function on the space itself. That "surfaces" might even be replaced by "curves" is the basis of recent interesting work on infinite-dimensional differentiation (as it was the basis of the very first work 250 years ago on that subject); the possibility of using curves as test figures may not be the result of the somewhat restrictive category of spaces considered, but rather of a more refined property of the basic codomains of functions, such as the line and circle. These are not only one-dimensional but even belong also to the Aufhebung of a still smaller level, since they are retracts of map-spaces of infinitesimal spaces.

The infinitesimal spaces, which contain the base topos in its non-Becoming aspect, are a crucial step toward determinate Becoming, but fall short of having among themselves enough connected objects, i.e. they do not in themselves constitute fully a "category of cohesive unifying Being." In examples the four adjoint functors relating their topos to the base topos coalesce into two (by the theorem that a finite-dimensional local algebra has a unique section of its residue field) and the infinitesimal spaces may well negate the largest essential subtopos of the ambient one which has that property. This level may be called "dimension $\in$ "; calling the levels (i.e. the subtoposes essential over the base) "dimensions" does not imply that they are linearly ordered nor that the Aufhebung process touches each of them. The infinitesimal spaces provide (in many ways) a good example of a non trivial unity-and-identity-of-opposites inside the ambient topos of Being: explicitly recognizing the *two* inclusions, as spaces which could be called infinitesimal and formal spaces respectively, may help clarify the confusing but powerful interplay between these two classes which are opposite but in themselves identical. The calculation of the $\in$ -skeleton and $\in$ -coskeleton, of a space which is neither, needs to be carried out, and also the calculation of the Aufhebung of dimension $\in$.

The idea behind the identification of the levels in a category of Being with dimensions is that a higher level is a more determinate general Becoming, that is, it contains spaces having in them possibly-more-varied information for determining processes. Thus one conjectures that dimX only depends on the category P(X) of particular Becoming associated to X (and not on the important structure sheaf which recalls for the little category the big environment in which it was born). In other words, if we have an equivalence of categories $P(X) \cong P(Y)$, then X,Y should belong to the same class of UIO levels within the category of Being in which they are objects. Suitable hypotheses to make this conjecture true should begin to clarify the relationships between the two suggested philosophical guides.

III. Why does the epsilon difference leave room for the triumph of geometry over narrow logicism? What might have led Grothendieck to propose his (still unpublished) program for "tame topology", wherein he arrived at roughly the same real analytic spaces as logicians working on a Tarski problem of "decidability"? It seems that all attempts to characterize continuity in a purely intensive logical way, such as the frame algebra, leave another kind of room in spite of their profound contribution to calculations - room for the obviously non-physical space-filling curves and nowhere-differentiable functions. Though we have been led to believe that this subjectively-generated Raum shows that our basic intuition of space is unreliable, still we have not been shown anything in the real world which could more than provisionally be modeled as a discrete infinity. Rather than such speculations about the unreliability of knowledge, it seems that still more serious work is needed, marshalling all the achievements of subjective logic *and* objective logic, of geometry *and* algebra to hone still more realistic models of continuous space. As several have suggested, a guide is to consider figures, i.e. maps to the space, as fundamental in determining it, with intensive quantities (i.e. maps from the space) being derived by naturality rather than the other way around; this does not metaphysically mean of course that the nature of the domains of the figures is not derived by a careful ascending interplay between all four of the mentioned subjects.

The above considerations are related, as suggested in a Como discussion, with old problems such as Fermat's. Diophantus probably considered natural numbers not in the abstract way which we habitually now do, but as born from actual objects. While the method of formally adjoining negatives and ascending to powerful cohomological calculations etc. leads to many results, we should not forget the objects themselves. Just as realizing cohomology classes by vector bundles via K-theory permitted powerful interplay between those calculations and directed manipulations of the objects by actual maps, so a similar possibility is opened by the Burnside rig of a distributive category, wherein polynomial equations satisfied by objects are revealed as specific structure on the objects themselves. For example, the equation $x = 1 + 2x$ arises from an object with a point and operated on by a 2-generator monoid, with an additional inverse map. But even the dangerous $x = 1 + x$ does not lead to unbridled infinity. While the study of linear equations on distributive categories is packed with surprising subtleties, higher-degree equations are also approachable with, for example, homogeneity retaining some of its usual properties when interpreted in this more demanding functorial manner. I was surprised to note that an isomorphism $x = 1 + x^2$ (leading to complex numbers as Euler characteristics if they don't collapse) always induces an isomorphism $x^7 = x$.

The rough similarity between Grothendieck's tame spaces and the finitely sub-analytic [FSA] objects considered in logic is in fact a major difference; the same sort of difference exists between real algebraic geometry and semi-algebraic sets, as well as between the ordinary continuous PL category and the polyhedral category of negative sets. All of these are quite different from categories in which countable coproduct decompositions are common. The difference within each of the three pairs mentioned may all exemplify a single general process which I'll call Aristotelian intervention; some such analysis seems also basic to attempts to hone a more physically realistic model for the programming of "digital" computers. Aristotle pointed out that the continuum is divisible but not infinitely divided. One can break a stick. Repeating that and all which it led to for 40,000 years has created a lot of indispensable chairs, buildings, etc. but has not changed the fundamental continuity of space; neither will billions of times dividing possible current-levels into "yes and no".

Given a map of coherent toposes, there is not only the usual induced topology but there are also topologies in which only the inverted maps between *coherent* objects are taken as covers. For example, to construct the generic solution of the equation $x = 1+3x+2x^2$, consider first the classifying topos for the free algebraic theory with one constant, three unary, and two binary operations; the free algebra on no generators determines a point of this topos. If we were to consider the full induced topology, our topos would collapse to that of discrete sets; a coherently induced topos however might not only satisfy the equation but have a non-trivial Burnside ring, whereas just because it lacks the metaphysical "minimality of the fixed point", it may provide a more physical model of potential lists and trees. We don't yet know which presentations of rigs can be Burnside-realized from distributive categories, because the very concreteness of the non-isomorphisms in the latter may give rise also to unexpected isomorphisms. We also don't know whether finitely subanalytic sets can be obtained via such a uniform procedure from some sort of continuously tame ones. Another possibility would be to take not the base topos but the infinitesimal one as the domain of the topos map which is used to induce the Aristotelian intervention - is it possible that even after such an explosion, functions could still have a well-defined derivative at every point? Certainly the resulting sites need not be Boolean; for example, consider a half-open interval x: it should satisfy $x = 2x$ but its endpoint is not a coproduct summand.

Naturally, models like the polyhedra constructed from below from real space are more satisfying than those constructed from above by classifying abstract algebras, but as usual the goal is to be approached by pushing hard from both sides.

If the general program proposed above is correct at least in rough outline, it would serve both the advancement and the dissemination of the subject to have it clearly worked out. As clearly formulated in Grassmann's introduction, only a good philosophical preamble can orient the student toward what kind of applications of a purely mathematical development he should look out for; that theory of pedagogy is at least as deserving of trial as the pragmatist theory of teaching only skills, which as we have seen did not achieve its goal.

In spite of temporary setbacks of all kinds, the many-sided and passionate advance of category theory has been on the whole remarkably steady. On the basis of all that work many questions of both fundamental and applied nature are now becoming clear, thus the future of our science is bright.

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