Categories of Space and of Quantity

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0. The ancient and honorable role of philosophy as a servant to the learning, development and use of scientific knowledge, though sadly underdeveloped since Grassmann, has been re-emerging from within the particular science of mathematics due to the latter's internal need; making this relationship more explicit (as well as further investigating the reasons for the decline) will, it is hoped, help to germinate the seeds of a brighter future for philosophy as well as help to guide the much wider learning of mathematics and hence of all the sciences.

1. The unity of interacting opposites "space vs. quantity", with the accompanying "general vs. particular" and the resulting division of variable quantity into the interacting opposites "extensive vs. intensive", is susceptible, with the aid of categories, functors, and natural transformations, of a formulation which is on the one hand precise enough to admit proved theorems and considerable technical development and yet is on the other hand general enough to admit incorporation of almost any specialized hypothesis. Readers armed with the mathematical definitions of basic category theory should be able to translate the discussion in this section into symbols and diagrams for calculations.

2. The role of space as an arena for quantitative "becoming" underlies the qualitative transformation of a spatial category into a homotopy category, on which extensive and intensive quantities reappear as homology and cohomology.

3. The understanding of an object in a spatial category can be approached through definite Moore-Postnikov levels; each of these levels constitutes a mathematically precise "unity and identity of opposites", and their ensemble bears features strongly reminiscent of Hegel's *Science of Logic*. This resemblance suggests many mathematical and philosophical problems which now seem susceptible of exact solution.

0. Renewed Progress in Philosophy Made Both Necessary and Possible by the Advance of Mathematics

In his Lyceum, Aristotle *used* philosophy to lend clarity, directedness, and unity to the investigation and study of particular sciences. The programs of Bacon and Leibniz and the important effort of Hegel continued this trend. One of the clearest applications of this outlook to mathematics is to be found in the neglected 1844 introduction by Grassmann to his theory of extensive quantities. Optimistic affirmations and applications of it are also to be found in Maxwell's 1871 program for the classification of physical quantities and in Heaviside's 1887 struggle for the proper role of theory in the practice of long-distance telephone-line construction. In the latter, Heaviside formulates what has also been my own attitude for the past thirty years: the fact that our knowledge will of course never be complete, and hence no general theory will be final, is no excuse for not using now the most general theory which science can support, and indeed for accuracy we must do so.

To students whose quest drives them in the above direction, the official bourgeois philosophy of the 20th century presents a near vacuum. This vacuum is the result of the Jamesian trend clearly analyzed by Lenin in 1908, but "popularized" by Carus, Mauthner, Dewey, Mussolini, Goebbels, etc. in order to create the current standard of truth in journalism and history; this trend led many philosophers to preoccupation with the flavors of the permutations of the thesis that no knowledge is actually possible. Naturally this 20th century vacuum has in particular tried to suck what it can of the soul of mathematics: a science student naively enrolling in a course styled "Foundations of Mathematics" is more likely to receive sermons about unknowability, based on some elementary abstract considerations about subjective infinity, than to receive the needed philosophical guide to a systematic understanding of the concrete richness of pure and applied mathematics as it has been and will be developed.

By contrast, mathematics in this century has not been at a standstill. As a result mathematicians at their work benches have been forced to fashion philosophical tools (along with those proofs of theorems which are allegedly their sole product), and to act as their own "Aristotles" and "Hegels" as they struggle with the dialectics of 'general' and 'particular' within their field. This is done in almost complete ignorance of dialectical materialism and often with understandable disdain for philosophy in general. It was struggle with a problem involving spheres and the relation between passage to the limit and the leap from quantity to quality which led Eilenberg and Mac Lane in the early 1940's to formulate the general mathematical theory of categories, functors, and natural transformations. Similarly, study of concrete problems in algebraic topology, functional analysis, complex analysis, and algebraic geometry in the 1950's led Kan and Grothendieck to formulate and use important further advances such as adjoint functors and abelian categories. And the past thirty years have not been devoid of progress: from the first international meeting on category theory in La Jolla, California in 1965 to the most recent one in Como, Italy in 1990, toposes, enriched categories, 2-categories, monads, parameterized categories (sometimes called "indexed"), synthetic differential geometry, simplicial homotopy, etc. have been refined and developed by over two hundred researchers with strong ties to nearly every area of mathematics. In particular all the now-traditional areas of subjective logic have been incorporated with improvement into this emerging system of objective logic.

It is my belief that in the next decade and in the next century the technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy. I can recall, for example, how my failure to learn the philosophical meanings of "form, substance, concept, organization" led to misinterpretation by readers of my 1964 paper on the category of sets and of my 1968 paper on adjointness in foundations; a more profound study of Hegel's *Wissenschaft der Logik* and of Grassmann's *Ausdehnungslehre* may suggest simplifications and qualitative improvements in the circle of ideas sketched below.

1. Distributive and Linear Categories; The Functoriality of Extensive and Intensive Quantities

A great many mathematical categories have both finite products and finite coproducts. (A product of an empty family is also known as a terminal object, and an empty coproduct as a coterminal or initial object). However, there are two special classes of categories defined by the validity of two special (mutually exclusive) relationships between product and coproduct. One of these may be called *distributive* categories, for these are defined by the requirement that the usual *distributive law* of arithmetic and algebra should hold for multiplication (=product) and addition (=coproduct) of objects, in the precise sense that the natural map from the appropriate sum of products to a product of sums should be an isomorphism; this includes as a special case that the product of any object by zero (=initial object) is zero. The other class of *linear* categories is defined by the requirement that products and coproducts *coincide;* more precisely, a coterminal object is also terminal in a linear category, which permits the definition of a natural map (="identity matrix") from the coproduct of any two objects to their product, and moreover this natural map is required to be an isomorphism. As pointed out by Mac Lane in 1950, in any linear category there is a unique commutative and associative addition operation on the *maps* with given domain and given codomain, and the composition operation distributes over this addition; thus linear categories are the general contexts in which the basic formalism of linear algebra can be interpreted.

All toposes are distributive. General categories of discrete sets, of continuous sets, of differentiable, measurable, or combinatorial spaces tend to be distributive, as do categories of non-linear dynamical systems. Given a particular space, there are categories of sheaves on it, of covering spaces of it, etc. which provide an expanded or restricted view of what happens in that particular space and are also distributive. Since both general ("gros") and particular ("petit") spatial categories are distributive categories, a useful philosophical determination would be the identification of "categories of space" with distributive categories. Since distributive categories such as that of the permutation representations of a group can often be seen to be isomorphic with spatial categories such as that of the covering spaces of a particular space having that group as fundamental group, the inverse identification has merit; it also permits to use geometrical methods to analyze categories of concepts or categories of recursive sets. For many purposes it is useful to "normalize" distributive categories by replacing them with the toposes they generate, permitting application of the higher-order internal logic of topos theory to the given distributive category; on the other hand many distributive categories are "smaller" than toposes and in particular have manageable Burnside rigs. Here by "rig" we mean a structure like a commutative ring except that it need not have negatives, and the name of Burnside was suggested by Dress to denote the process of abstraction (exploited recently by Schanuel) which Cantor learned from Steiner: the isomorphism classes of objects from a given distributive category form a rig when multiplied and added using product and coproduct; the algebra of this Burnside rig partly reflects the properties of the category and also partly measures the spaces in it in a way which (as suggested by Mayberry) gives deeper significance to the statement attributed to Pythagoras: "Each thing is number". Still in need of further clarification is the contrast within the class of distributive categories between the "gros" (general category of spaces of a certain kind) and the "petit" (category of variable sets over a particular space); this distinction (a qualitative one, not one of size) has been illuminated by Grothendieck and Dubuc and, I hope, by my 1986 Bogotá paper [14]; these show the importance of the ways in which an object in a "gros" category can give rise to a "petit" category, and the additional "structure sheaf" in the "petit" category which reflects its origin in the "gros" environment.

The category of real "vector spaces", the category of abelian groups, the category of topological vector spaces and the category of bornological vector spaces are all linear categories. So are the category of projective modules over any particular rig and the category of vector bundles over any particular space. In the last example, the vector bundles (=objects) themselves are *kinds* of variable quantities over the space, and the *maps* between these are particular variable quantities over the space. Thus "categories of quantity" will be tentatively identified with linear categories. Abelian AB5 categories are special linear categories having further "exactness" properties; again "normalization" may be useful, even within functional analysis. For abelian categories and many others, the Mac Lane addition of maps is actually an abelian group, that is, each map has a negative. However, for some other linear categories addition is actually idempotent (and hence could not have negatives in this algebraic sense); this occurs in logic (in the narrow sense) where the quantities are variable truth values (reflecting "relations"), and in geometry when quantities are (variable) dimensions and the multiplication is *not* idempotent.

What is a space and how can quantities vary over a space? We have suggested above that, formally, a space is either a "petit" distributive category or an object in a "gros" distributive category. But as spaces actually arise and are used in mathematical science, they have two main general conceptual features: first they serve as an arena for "becoming" (there are spaces of states as well as spaces of locations) and secondly they serve as domains for variable quantity. These two aspects of space need to be expressed in as general a mathematical form as possible: in section 2, I will return to "becoming" and one of its roles in mathematics, but in this section 1 concentrate on the relation between space and variable quantity.

Broadly speaking there are two kinds of variable quantity, the extensive and the intensive. Again speaking broadly, the extensive quantities are "quantity of space" and the intensive quantities are "ratios" between extensive ones. For example, mass and volume are extensive (measures), while density is intensive (function). Although Maxwell managed to get extensive quantities accepted within the particular science of thermodynamics, and although Grassmann demonstrated their importance in geometry, there is still a reluctance to give them status equal to that of functions and differential forms; in particular the use of the absurd terminology "generalized function" for such distributions as the derivative of the Dirac measure has created a lot of confusion, for as Courant in effect observed, they are not intensive quantities, generalized or not. "Generalized measure" would have been a better description of distributions; to show that a distribution "is a function" involves finding a density for it relative to a "fixed" reference measure, but only in special non-invariant circumstances do the latter exist.

Broadly, a "type of extensive quantity" is a covariant coproduct-preserving functor from a distributive category to a linear category. The last condition reflects the idea that if a space is a sum of two smaller spaces, then a distribution of the given type on it should be determined by an arbitrary pair of distributions, one on each of the smaller spaces, while by the defining property of a linear category, "pairs" are equally well expressed in terms of coproducts in the codomain of our functor. The covariant functoriality has itself non-trivial consequences: the value of the functor at the terminal space may be considered to consist of *constant* quantities of the given type, and the value of the functor at a given space to consist of the extensive quantities of the given type which are *variable* over that space; since any given space has a unique map to the terminal space, the functor induces a map in the linear category which assigns to each variable extensive quantity its *total*, which is a constant. For example, the quantity of smoke now in my room varies over the room, but in particular has a *total*. On the other hand a map *from* the terminal space *to* a given space is a *point* of that space; thus the functor assigns to such a point a linear map which to any constant weight of the given type assigns the Dirac measure of that weight which is supported on that point. For a more particular example of the covariant functoriality in which neither domain nor codomain of the inducing map reduces to the terminal space, consider the following definition of the term *sojourn:* the extensive quantity-type is time(-difference) and there are two spaces, one representing a time interval of, for example, July and the other for example, the continent of Europe. On the first space there is a particular extensive quantity of this type known as duration. A particular journey might be a map (in an appropriate distributive category) from the first space to the second, hence via the functor the journey acts on the duration to produce on the continent a variable extensive quantity known as the sojourn (in each given part of the continent) of my journey. As another example, if I project my room onto the floor, the quantity of smoke is transformed into the quantity of smoke *over* the floor.

A further determination is suggested by the idea "space of quantity" which lies at the base of (not only cartesian coordinatizing but also) calculus of variations and functional analysis: the variable quantities (extensive or intensive) of a given type over a given space should themselves form a space (often infinite-dimensional) which contains its own processes of "becoming" (continuous, differentiable, etc.) and is itself the domain of further variable quantities. This idea can be realized as follows: over a given distributive category of spaces, consider the linear category of all spaces equipped with given additions and all maps which preserve these; the forgetting functor from the latter to the former expresses in a general way that these quantity-types "are" spaces. But then in particular an extensive quantity-type from the distributive category to *this* linear category can be subjected to the further requirement that it be enriched (or strong) in the sense of enriched category theory, i.e. roughly that as a functor it be concordant with "becoming" (parameterization).

By contrast an intensive quantity-type is a *contravariant* functor, taking coproducts to products, from a distributive category, but now a functor whose values have a multiplicative structure as well as an additive structure. Frequently the values of an intensive type are construed to be rigs, such as the ring of continuous or smooth functions or the lattice of propositional functions on the various spaces in the distributive category, with the functoriality given by substitution; however, since we also need to consider vector- and tensor-valued "functions", it is more adequate to consider that a typical value of an intensive quantity-type is itself a linear category, with composition in the latter being the multiplicative structure and with each spatial map inducing via the type a linear functor (in the opposite direction) between the two "petit" categories of intensive quantities on the domain and codomain spaces of the map. From the latter point of view the rigs are just endomap objects of certain preferred objects in these intensive categories, and in some examples (such as the analytic, though not the differentiable, study of projective space), knowledge of the rigs may not suffice to determine the intensive categories.

To exemplify the contravariant functoriality, the terminal map from a given space induces the "inclusion" of constant quantities of the given type as special "variable" intensive quantities on the space, while a given point of the space induces the *evaluation* at that point of any intensive quantity (caution: in general an intensive quantity may not be determined by the ensemble of its values at points); a particular journey of a month through a continent induces a transformation of any intensive quantity on the continent (such as the frequency with which a given language can be heard) into an intensive quantity varying over the month.

Again by specializing to the linear objects *in* the given distributive category as possible map-objects for the intensive categories assigned to each space, the important "space of quantity" idea, as well as a further enrichment requirement on the types, can also be realized for *intensive* quantities.

Moreover, if the distributive category is actually "cartesian-closed" (so has a "space of maps" between any two spaces, satisfying the objective relations which were used since the first days of the calculus of variations and which in this century were subjectively codified as "lambda-calculus") then the further important idea of the possible *representability* of components of an intensive quantity-type comes into play. Namely, the represented intensive quantity-type is defined to have as objects always the linear spaces in the distributive category itself, but each given space is defined to have as the map-objects of the corresponding intensive category the space of all maps from the given space to the spaces of linear maps between given linear spaces, the latter being the "representors"; an intensive quantity type is called representable if it is equivalent to a full part of this represented one. For example, the usual ring of smooth functions is representable when the constant scalars form a smooth space, and the lattice of propositional functions is representable when truth-values form a space (as they do in a topos).

It should be pointed out that there is a second doctrine of extensive/intensive quantities which agrees with the above when only "compact" spaces are considered, but which in general permits only "proper" spatial maps to induce (co-and contra-variantly) maps of quantities. Since they admit "totals", the extensive quantities which I described above should perhaps be thought of as being restricted to have "compact support", while the intensive quantities are "unrestricted" and thus might be representable, both of these features being compatible with my requirement of functoriality on *arbitrary* spatial maps in the distributive category. By contrast, the second "proper" doctrine is useful when considering "unrestricted" *extensive* quantities (such as area on the whole plane) but must correspondingly impose "compact support" restrictions on the *intensive* quantities, making the latter *non*-representable. These remarks presuppose the *relation* between extensive and intensive quantities, to which I will now turn.

The common spatial base of extensive and intensive quantities also supports the relation between the two, which is that the intensives act on the extensives. For example, a particular density function acts on a particular volume distribution to produce a resulting mass distribution. Thus it should be possible to "multiply" a given extensive quantity on a certain space by an intensive quantity (of appropriate type) on the same space to produce another extensive quantity on the same space. The definite *integral* of the intensive quantity "with respect to" the first extensive quantity is defined to be the *total* of this second resulting extensive quantity. This action (or "multiplication") of the contravariant on the covariant satisfies bilinearity and also satisfies, with respect to the multiplicative structure within the intensive quantities and along any inducing spatial map, an extremely important strong homogeneity condition which so far has carried different names in different fields: in algebraic topology this homogeneity is called the "projection formula", in group representation theory it lies at the base of "Frobenius reciprocity", in quantum mechanics it is called "covariance" or the "canonical commutation relation", while in subjective logic it is often submerged into a side condition on variables for the validity of the rule of inference for existential quantification when applied to a conjunction.

It is in terms of such "action" (or "multiplication") of intensive quantities on extensive quantities that the role of the former as "ratios" of the latter must be understood. As in the study of rational functions and in the definition of derivative, algebra recognizes that multiplication is fundamental whereas "ratio" is an inverse process; while the simple prescription "you can't divide by zero" may suffice for constant quantities, its ramifications for variable quantities are fraught with particularity, as reflected in even the purely algebraic "localization" constructions. For example, a given mass or charge distribution may not admit a density, with respect to volume, and not only the existence but also the uniqueness of such ratios may require serious study in particular situations, even though the multiplication which they invert is "everywhere" well-defined; the famous Radon-Nikodym theorem gives conditions for this in a specific context.

How can systems of extensive and intensive quantities, with action of the latter on the former, be realized on various distributive categories which mathematically arise? As mentioned above, the intensive quantities are often representable (indeed more often than commonly noticed, for example differential forms can be represented via the "fractional exponentiation" which exists in certain gros toposes). An important class of extensive quantities can be identified with the (smooth linear) *functionals* (with codomain a fixed linear space such as that of constant scalars) on the given intensive quantities, i.e. a distribution may sometimes be determined by the ensemble of all definite integrals (with respect to it) of all appropriate intensive quantities. This identification, supported in a particular context by the classical Riesz representation theorem (and in the homotopical context of section 2 below, by the universal coefficient theorem), contributed to the flourishing of functional analysis, but perhaps also distracted attention from the fact that extensive quantities are at least as basic as the intensive ones. At any rate, the fundamental projection formula/canonical commutation relation is automatic for those extensive quantities which can be identified as functionals on the intensive ones; here the action is *defined* in terms of the integral of the multiplication of intensive quantities.

This automatic validity of the fundamental formula holds also for a certain "opposite" situation in which a concept of intensive quantity can be defined to consist of transformations on given extensive concepts. More precisely, recall that I suggested above a general definition of extensive quantity type on a given distributive category as an enriched additive covariant functor from the given distributive category to the linear spaces in it. Given two such functors, we can consider *natural transformations* from one to the other, which thus can tautologously "multiply" extensive quantities of the first type to yield extensive quantities of the second type. Such natural transformations, however, are *constant* intensive quantities (i.e. "varying" only over the terminal space) since they operate over the whole distributive category. But the idea of natural transformation also includes all *variable* intensive quantities over some given space (and between two extensive functors), if we only make the following modification. An extremely useful construction, first emphasized by Grothendieck around 1960 (although it occurs already in Eilenberg and Mac Lane's original paper), associates a new category to any given object in a given category by considering as new objects all the *maps* with codomain the given object, and as new maps all the commutative triangles between these; this construction, a special case of the ill-named "comma category", has manifold applications revolving around the idea that both a *part* (with "multiplicity") of the given space as well as a *family* of spaces ("the fibers") smoothly parameterized by the given space are themselves objects in a new category; borrowing from Grothendieck, we may for short call this category the "gros" category of the given space (the "gros" category of the *terminal* space reducing to the given distributive category). Often a distributive category is in fact *locally* distributive, in the sense that for each space in it the associated "gros" category is again distributive. (The "petit" category of a space is usually a certain full subcategory of its "gros" category). A map between two spaces obviously induces by composition a coproduct preserving functor from the "gros" category of the first to the "gros" category of the second; in particular, the "gros" category of a space thus has a forgetting functor to the original distributive category of spaces. Composing this forgetting functor with two given extensive types, an intensive quantity varying over the given space may then be defined to be any natural transformation between the resulting composite functors. Thus according to this point of view, in the intensive category associated to a space, not only are the *maps* identified with intensive quantities varying over the space, but the *objects* are (or arise from) the types of extensive quantity which the whole category of spaces supports.

The most fundamental measure of a thing is the thing itself. If we replace "thing" by "object" (for example object in a category of spaces), then "itself" may be usefully identified with the Pythagoras-Steiner-Burnside abstraction process discussed earlier: that is, isomorphic objects are identified, but all other maps are temporarily neglected. This obviously depends on what category the object is in, and the maps still play an important role in constructing and comparing new categories upon which the same abstraction process can be performed, notably the "gros" or "comma" categories of given spaces (as discussed above) and various "petit", "proper", "covering", "subobject" etc. subcategories of these. Moreover, in any locally distributive category there is for each map a "pullback" functor between the associated pair of "gros" categories, right adjoint to the obvious composition/forgetting functor previously mentioned. Thus, given a class of objects closed under coproduct (for example the class of finite, or discrete, or compact objects, or of the objects of fixed dimension, or intersections of these classes, etc) one can define a corresponding extensive quantity-type by assigning to each space (the abstraction of) the part of the "gros" category of that space which consists of those maps whose domains are in the class; this is obviously covariantly functorial via the composition/forgetting procedure. Given two such classes of objects, an intensive quantity from the one to the other, varying over a given space, can be defined to be (the abstraction of) any object of the "gros" category of the space which is *proper* in the sense that pulling back by it takes extensives of the one class into extensives of the other. Both the contravariant functoriality of these intensives as well as (tautologously) their action on such extensives is given by pullback, and the projection formula/CCR results from simple general lemmas about composition and pullback valid in any category. This *concrete* doctrine of quantity is explicitly or implicitly used in many branches of geometry, and I suspect that its direct use in many applications would be easier than translating everything into numbers (I recall a restaurant in New York in which customers, cooks, waiters, and the cashier may speak different languages, yet rapid operation is achieved without any written orders nor bills by simply stacking used dishes according to shape). One of the unsolved problems of the foundations of mathematics is to explain how and where the usual smooth distributions and functions of analysis can be obtained in this concrete mode.

As already the Grassmann brothers understood, the basic subject-matter of narrow-sense logic is quantities which are additively idempotent. The *intensive* aspect of this has been much studied, and is (at least fundamentally) concrete in the above sense, corresponding to parts without multiplicity (i.e. to subobjects); indeed one of the two basic axioms of topos theory is that subobjects are representable by (indicator maps to) the truth-value space. On the other hand the great variety of useful *extensive* logic has been little studied (at least *as* logic). In practice logic is not really a starting-point but rather the study of *supports and roots* of non-idempotent quantity: for example, the inhabited *part* of the world is the part where population exists, yet population (unlike the indicator of the part) is a non-idempotent quantity; distributions have supports and a pair of functions determines the ("root-") space of their agreement as well as the ("open") subspace of their disagreement. While the Dedekind definition of a real intensive quantity as an ensemble of answers to yes-no questions has many uses, we should not let pragmatism blind us to the fact that a procedure for coming to know the quantity is by no means identical with the objectively operating quantity itself. The (still to be studied) extensive logic should be the codomain of an adequate general theory of the supports of extensive quantities, a theory accounting for certain rules of inference as reflections of the commutation relations for variable quantities; such relationships are studied in the branch of algebraic geometry known as intersection theory, but raising certain aspects of the latter to the level of philosophy should help to make them more approachable and also to suggest in what way they might be applied to other distributive categories.

It may be that, to accord more accurately with historical philosophical terminology, all the above occurrences of the word "quantity" should be replaced by "number", with the former being reserved for use in conjunction with the "affine" categories whose study has recently been revived by Schanuel, Carboni, Faro, Thiébaud and others; Grassmann seems to insist that numbers are *differences* of quantities (as for example work is a difference of energies, and duration a difference of instants), and further understanding of affine categories may reveal them as an objective basis of the link between distributive and linear categories. There are moreover "non-commutative affine objects" known as "symmetric spaces" which include not only Lie groups, but also spheres, but whose intrinsic categorical property and role has been little explored.

2. Homotopy Negates yet Retains Spatiality

The role of space as arena of "becoming" has as one consequence a quite specific form of the transformation of quantitative into qualitative; the seemingly endless elaboration of varied cohomology theories is not merely some expression of mathematicians' fanatical fascination for fashion, but flows from the necessity of that transformation.

One of the main features which distinguish the general "gros" spatial categories from the particular "petit" ones is the presence of spaces which can act as internal parameterizers of "becoming". Formally, the essential properties of such a parameterizer space are that it is connected and strictly bipointed. Connectedness means that the space is not the coproduct of smaller spaces; when the category has a subcategory of "discrete" spaces, the inclusion functor having a left adjoint "components" functor, then connectedness of a space means that its space of components is terminal. Strict bipointedness means that (at least) two points (maps from the terminal space) of the space can be distinguished, where "distinguished" is taken in the strong sense that the two points are disjoint as subobjects, or in other words their "equalizer" is initial. (These definitions are often usefully extended from objects to "cylinders", i.e. to maps (with not-necessarily-terminal codomains) with a pair of common "sections" (generalizing "points")).

In order to maintain the rather heroic avoidance, which this paper has so far managed, of the traditional use of symbols to multiply the availability of pronouns, I will refer to the points of such a strictly bipointed connected space as "instants", without implying that that space is "one-dimensional" nor any further analysis of time. Note that, depending on the nature of the ambient "gros" distributive category, connectedness need not imply that the object is infinite; for the planning of activities and of calculations in the continuous material world, finite combinatorial models of the latter are necessary, and such models may in themselves constitute an appropriate category, the category of "simplicial sets" being a widely-used example.

Now a specific process of "becoming" in a certain space will be (accompanied by) a specific map to that space from a connected strictly bi-pointed space, whereby in particular the point to which one distinguished instant is mapped "becomes" the point to which the other distinguished instant is mapped. In particular, one map between two given spaces can "become" a second such map, as is explained by the usual definition of "homotopy" between the two maps, or equivalently (in the Hurewicz spirit), using if necessary the technique of embedding in presheaf categories to construe any distributive category as (part of) a cartesian-closed one, by applying directly the above account of "becoming" to points in the appropriate map-space. Obviously a very important application (of such internalization to a spatial category of the notion of "becoming") is the detailed study of dynamical processes themselves, bringing to bear the rich mathematical content which the category may have. However, in this section I will concentrate on the qualitative structure which remains after all such connected processes of "becoming" are imagined completed, that is, after any two maps which can possibly become one another are regarded as identical.

The traditional description of quantity as that which can be increased and decreased leads one to define a space as "quantitative" if it admits an action of a connected strictly bipointed object, wherein one of the two distinguished instants acts as the identity on the space, whereas the other acts as a constant; thus the whole space can be "decreased to a point". This is a much stronger requirement than connectedness of the space and is usually called *contractibility*. This use of "quantitative" is not unrelated to the use of "quantity" in section 1, since representing objects for intensive quantities are often contractible.

With a "gros" spatial category it is usually possible to associate a new category called its *homotopy category*, in which homotopic maps become equal and contractible spaces all become isomorphic to the terminal space; in general two spaces which become isomorphic in the homotopy category are said to have the same homotopy type. For this association to exist the composition of homotopy-classes of maps must be well-defined, which in turn rests on an appropriate compatibility between connectedness and categorical product; in particular the product of two connected spaces should be connected. The latter is almost never true for "petit" distributive categories. In case the category is cartesian closed and has a components functor, the appropriate compatibility is assured if the components functor preserves products. To any such case Hurewicz's definition of the homotopy category, as the one whose map-spaces are the component spaces of the original map-spaces, can immediately be generalized, and indeed also extended to any category enriched in the given spatial category, such as pointed spaces, spaces with given dynamical actions, etc. yielding corresponding new qualitative categories which are enriched in the homotopy category. Using the product-preservation of the components functor and the fact that composition in a cartesian-closed category can be internally construed as itself a map whose domain is a product (of two map-spaces), it is easy to see that Hurewicz's definition supports a unique reasonable definition of composition of the maps in the homotopy category.

Now the main point which I wish to make is that essentially the whole account of space vs. quantity and extensive vs. intensive quantity given in section 1 reproduces itself at the qualitative level of the homotopy category. The latter is itself again a distributive category, cartesian closed if the original spatial category was. Indeed more precisely, the homotopy-type functor connecting the two actually preserves products, coproducts, and the map-space construction. On the other hand the homotopy category is *not* locally distributive: the passage to the parametric homotopy category of the "gros" category *of* a given parameter space seems to involve a further qualitative leap, not as passive as the corresponding passage in the quantitative context.

Although obtained by nullifying "quantitative" spaces, the homotopy category still admits extensive measurements of its objects, the most basic ones being the number of holes of given dimensions. The extensive quantity-types here are usually called homology theories. Dually, the intensive quantity-types are cohomology theories, enjoying the features of contravariance, multiplicativity ("cup" product), and action as "ratios" of homology quantities. By a celebrated theorem, cohomology is often *representable* on the homotopy category, by objects known for their discoverers as Eilenberg-Mac Lane spaces. There is a strong tendency for basic homology and cohomology quantities to be (approximated by) linear functionals on each other. A new feature (probably distinguishing homotopy categories from the other distributive categories which do contain "becoming"-parameterizers and "quantitative" objects, although an axiomatic definition is unclear to me) is the appearance of *homotopy groups*, extensive quantity-types finer than homology and (co-) *representable* (by spheres). Note that the definition of "point" when applied in a homotopy category in fact means "component".

3. "Unity and Identity of Opposites" as a Specific Mathematical Structure; Philosophical Dimension

Not only should considerations of the above sort provide a useful guide to the learning and application of mathematics, but also the investigation of a given spatial category can be partly guided by philosophical principles. One of these is described, in conjunction with a particular application, in my paper "Display of graphics and their applications, as exemplified by 2-categories and the Hegelian 'Taco'".

Namely, within the system of subcategories of the category to be investigated, one can find a structure of ascending richness which closely parallels that of Hegel's *Science of Logic*, with each object to be investigated having its reflection and coreflection into each of the ascending levels. Here a level is formally defined as a functor from the given category (to a "smaller" one) which has both left and right adjoint sections; these sections are then the full inclusions of two subcategories which *in themselves* are "identical" (to the smaller category) but which as *subcategories* are "opposite" in the perfectly precise sense given by the adjointness, and the two composite idempotent functors resulting on the given category provide (via the adjunction maps) the particular reflection and coreflection in this level of any given space. In combinatorial topology such a level is exemplified by all spaces of dimension less than n, with the idempotent functors being called n-skeleton and n-coskeleton; in other cases the "dimensions" naming the levels may have a structure more (or less) rich than that of just natural numbers. Dimension "minus infinity" has the initial object and the terminal object as its two inclusion functors (in themselves, both are identical with the terminal category, but in the spatial category they are opposite); it seems to me that there is good reason to identify the initial and terminal objects with Hegel's "non-being" and (pure) "being" respectively. While the relation of one level's being lower than another (and hence of one "dimension's" being less than or equal to another) can be defined in an obvious categorical way, the special nature of the levels subjects them to the further relation of being "qualitatively lower": namely, one level is qualitatively lower than another level if both its left and right adjoint inclusions are subsumed under the single *right* adjoint inclusion of the higher level. In many examples there is an Aufhebung or "jump": for a given level there is a smallest level qualitatively higher than it.

The Aufhebung of dimension "minus infinity" is in many cases "dimension 0", the left adjoint inclusion providing the discrete spaces of "non-becoming", the opposite codiscrete spaces forming an identical category-in-itself which is however now included as the chaotic "pure becoming" in which any point can become any other point in a unique trivial way. Both initial and terminal objects are codiscrete. This level zero (in itself) is often very similar to the category of abstract sets, although (for example in Galois theory) it may not be exactly the same; as I tried to explain in my 1989 Cambridge lectures, the double nature of its inclusion into mathematics may help to resolve problems of distinguishability vs. indistinguishability which have plagued interpretation of Cantor's description of the abstraction process, (and hence obscured his definition of cardinals). This discrete/codiscrete level is often special in the further respect that its left (discrete) inclusion functor has a further left adjoint, the "components" functor for the whole category (which, as discussed above, should further preserve finite products).

The Aufhebung of dimension zero strongly deserves to be called dimension one: its equivalent characterization, as the smallest level such that any space has the same components as the skeleton (at that level) of the space, has the clear philosophical meaning that if a point (or figure) can become another one, then it can do so along some 1-dimensional process of "becoming". Here dimensionality *of a space* (such as a parameterizer) is defined negatively in terms of skeleta (rather than "positively" in terms of coskeleta which are typically infinite dimensional).

For the levels qualitatively higher than zero, the *right* adjoint inclusion also preserves *co*-products, a very special situation even for topos theory. In a topos having a "becoming"-parameterizer, the truth-value object itself is contractible (as pointed out by Grothendieck), permitting "true" to become "false" in a way overlooked by classical logic; hence the 1-skeleton of the truth-value space presents itself as a canonical (though perhaps not adequate) parameterizer of "becoming" or "interval". These suggest only a few of the many open problems, involving calculation of the many examples, which need to be elaborated in order to clarify the usefulness of these particular concrete interpretations of the dialectical method of investigation. We very much need the assistance of interested philosophers and mathematicians.

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