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Qualitative Distinctions Between Some Toposes of Generalized Graphs

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I have divided this paper into three sections as follows: Section I sketches, in a somewhat impressionistic way, the general conceptual context which I intend to exemplify in section III; section II reviews some of the basic constructions on presheaf categories which are implicit in nearly all branches of mathematics, with some examples which it is hoped will appeal to computer scientists; section III describes three sequences of "simplest possible" examples of some of the basic phenomena, pointing out some of the peculiarities which arise even within these sequences.

I. This section should become clearer on a second reading, after having studied the rest of the paper.

"Being is doing", and hence particular being is known (at least partly) by what it can do. If B is an object in a "gros" topos $\mathcal{B}$ of cohesive active sets, what it can do is to continuously parameterize and dynamically act on mathematical structures. The zeroth form of mathematical structure is the abstract sets which form a Boolean topos and which can be realized in two opposite ways (chaotic and discrete) $\mathcal{S} \leftrightarrows \mathcal{B}$ as very special extreme cases of "cohesive active" sets. The category $\mathcal{S}(B)$ of all possible ways that B can parameterize and act on abstract sets is again a topos, but of a qualitatively different sort called "petit" by Grothendieck (typically both the gros $\mathcal{B}$ and all the petit $\mathcal{S}(B)$ are proper classes from the mid-century "set-theoretic" point of view; it is not cardinality but another quality which distinguishes the two). There is a geometric morphism of toposes $\mathcal{B}/B \longrightarrow \mathcal{S}(B)$ from the non-petit "comma category", which already exhibits the qualitative difference even in case $B = 1$, the terminal object of $\mathcal{B}$: for $\mathcal{B}/1 = \mathcal{B}$ but $\mathcal{S}(1) = \mathcal{S}$, and the geometric morphism $\mathcal{B} \longrightarrow \mathcal{S}$ is the functor whose right adjoint is the inclusion

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as chaotics and whose left adjoint is the inclusion as discretes; moreover, in this case the discrete inclusion has a further left adjoint $\pi_0 : \mathcal{B} \longrightarrow \mathcal{S}$ which assigns to each cohesive/active set B the corresponding abstract set of its components/orbits with the adjunction morphism $B \longrightarrow \pi_0(B)$ being the universal map to a discrete space; moreover, the truth value object $\Omega_{\mathcal{B}}$ of $\mathcal{B}$ is *connected* $\pi_0(\Omega) = 1$, hence in particular $\Omega_{\mathcal{B}} \neq 1 + 1$ nor any other sum so that $\mathcal{B}$ cannot be Boolean, whereas $\Omega_{\mathcal{S}} = 1 + 1$ (the earmark of Booleanness) and π_0 (being a left adjoint) preserves sums so that in a positive sense $\mathcal{B} \to \mathcal{S}$ is far from an equivalence. In most determinations of $\mathcal{B}$ which I have considered, $\mathcal{S}$ is in fact determined by $\mathcal{B}$ in that the codiscrete inclusion $\mathcal{S} \hookrightarrow \mathcal{B}$ is characterized as the smallest subtopos of $\mathcal{B}$ which contains the empty object 0 of $\mathcal{B}$. Moreover, it seems essential that π_0 has a tendency to preserve finite products for gros $\mathcal{B}$. Thus what I have said puts many stringent restrictions on the topos $\mathcal{B}$.

I have used above the term "space" as short for cohesive/active set; already in Grassmann it was clear that space is generated by, and lays the foundation for, motion and hence that general spaces have aspects of both. These two aspects can be illustrated in somewhat pure form as follows: If B is essentially determined by a poset of regions in some space, ordered by inclusion (pure cohesiveness in one of its simpler manifestations) it can still act in the following sort of way (presheaf): Let UX be the (abstract) set of all X-valued smooth functions on U for any region U of B: then an inclusion $U' \subset U$ in B acts by restriction to give $UX \longrightarrow U'X$, the totality of sets of functions and restriction-actions determining a single object $\underline{X}$ of $\mathcal{S}(B)$. At the opposite extreme (pure activeness) we might consider a B which is determined by a *suitable monoid* (for example the group of all rigid motions of physical space under composition, or the set of non-negative time durations under addition) and take for $\mathcal{S}(B)$ the category of all *right* actions of B on abstract sets. But already here the "pure" activeness passes over into cohesiveness: if the monoid B satisfies the cancellation property which says that *left* multiplication by any element is a monomorphism, and if $T_B \in \mathcal{S}(B)$ denotes the regular representation [the Yoneda-Grothendieck-Dedekind-Cayley embedding gives a *single* object in the case of a monoid] then the topos $\mathcal{S}(B)/T_B$ (whose objects are objects of $\mathcal{S}(B)$ equipped with a morphism to T_B and whose morphisms are commutative triangles over T_B in $\mathcal{S}(B)$) can also be expressed as $\mathcal{S}(B)/T_B \cong \mathcal{S}(B/*)$ where $*$ is the unique object of B, but $B/*$ is in such a cancellative case a *poset.* Thus in such a case the category $\mathcal{S}(B)/T_B$ of "non-autonomous dynamics" derived from pure action $\mathcal{S}(B)$ turns out to be pure cohesiveness. [Caution: If B is a group then $\mathcal{S}(B)/T_B \approx \mathcal{S}$ since then the poset $B/*$ is codiscrete.] Of course already the "orbit set" functor $\mathcal{S}(B) \longrightarrow \mathcal{S}$ reveals some of the cohesiveness induced by dynamical action: if two elements of a state space $X \in \mathcal{S}(B)$ can be moved into a common element by some acts of B, they thereby "stick together" in one sense. If a monoid B_1 acts on a pure space B_0 then there will be an induced action of B_1

on the poset of subregions of B_0 and the resulting semidirect product $B_0 \ltimes B_1$ is a genuinely mixed cohesive/active set, leading to $\mathcal{S}(B_0 \ltimes B_1)$ etc.

Many of the kinds of mathematical structures which need to be parameterized and acted on by B are commonly understood in their constant form as conditioned diagrams in a category $\mathcal{S}$ of abstract sets, for example a group $1 \longrightarrow \overset{\circlearrowleft}{S} \longleftarrow S \times S$, or a poset $E \rightrightarrows S$ (where the induced single $E \hookrightarrow S \times S$ is monomorphic and contains $S \longrightarrow E$ as the diagonal and *is* the "order relation"), etc. Hence the *variable* groups, posets etc. may to a large extent be treated as exactly similar diagrams *in* the topos $\mathcal{S}(B)$ of *variable* abstract sets; that is

$$B\text{-variable } A\text{-structured sets}$$

$$= A\text{-structured } B\text{-variable sets}$$

often holds, where A is any "theory" of quite a general kind. Here B-variable sets $= \mathcal{S}(B) =$ the topos of abstract sets parameterized and acted on by B.

One of my aims will be to bring out properties of the "petit" toposes $\mathcal{S}(B)$ which will distinguish them in a positive way from the "gros" toposes such as $\mathcal{B}$. A typical $\mathcal{S}(B)$, unlike the special case $\mathcal{S}(1) = \mathcal{S}$, will *not* be Boolean in its equational logic

$$\Omega \times \Omega \overset{\wedge}{\underset{\Rightarrow}{\overset{\vee}{\rightleftarrows}}} \Omega,$$

but Heyting like $\mathcal{B}$; on the other hand its *object* Ω of truth values will almost never be connected, unlike $\mathcal{B}$.

Why is it important that a gros topos $\mathcal{E}$ have a *finite product preserving* components functor $\pi_0 \dashv \text{discrete} \dashv \mathcal{E}(1, -)$? For one thing it means that the Burnside ring of $\mathcal{E}$ is the monoid ring (of measures under convolution) of the monoid of finite connected objects under cartesian product, avoiding the series of structure constants involved in a petit topos such as G-sets for a group(oid) G. More importantly perhaps, it means that for any category $\mathcal{A}$ enriched in $\mathcal{E}$, (that is a category whose homs are "spaces" $\mathcal{A}(A', A) \in \mathcal{E}$ rather than abstract sets) such as $\mathcal{A} = \mathcal{E}$ itself, we can begin the qualitative "homotopical" classification of the objects of $\mathcal{A}$ without fractions by defining an associated abstract category $\pi\mathcal{A}$ via $(\pi\mathcal{A})(A', A) = \pi_0\mathcal{A}(A', A)$, being assured by the product-preserving property of π_0 that these "homotopy-classes" of $\mathcal{A}$-maps can still be composed. Roughly, if $\mathcal{A}$ is the study of specific processes $a_0 \xrightarrow{\alpha} a_1$ in objects A, then $\pi\mathcal{A}$ is the study of the higher ramifications of the ensemble of conditions "there exists a process $a_0 \dashrightarrow a_1$ of the kind allowed by A"; here by "process" we mean one which proceeds by one simple law at least to the extent that if for example

$$a_k = \alpha\partial_k,$$
$$p\partial_k = 1_{A'}$$

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, shift left, "p"] & A \\
A' \arrow[u, shift left, "\delta_0"] \arrow[u, shift right, "\delta_1"'] \arrow[ur, shift left, "a_0"] \arrow[ur, shift right, "a_1"'] &
\end{tikzcd}$$

with A' a terminal object in $\mathcal{A}$, we want L also to be "contractible" in $\mathcal{A}$ in the sense that L becomes terminal in $\pi\mathcal{A}$. Such considerations are so compelling that one is driven to believe that if some approximation $\mathcal{E}$ to the notion of "continuous map" fails to have product-preserving π_0, then $\mathcal{E}$ should be changed (as it is often done to achieve cartesian-closure).

II. The next few paragraphs may be simply skimmed (to assimilate my notation) by any reader already familiar with the basics of category theory, of which they are mainly a review.

The examples which we wish to consider are all presheaf toposes, so that Grothendieck topologies do not play a major role although coverings may. Thus for any small abstract category $\mathbf{C}$ (i.e. one whose total set of morphisms constitutes a single object in the category $\mathcal{S}$ of abstract sets, and whose domain, codomain, identity specification, and composition operations are maps in $\mathcal{S}$) we will be considering the category $\mathcal{S}^{\mathbf{C}^{op}}$ of all contravariant functors from $\mathbf{C}$ to $\mathcal{S}$ and all natural transformations between these; an object X of $\mathcal{S}^{\mathbf{C}^{op}}$ may equally well be considered as a *right action* of $\mathbf{C}$ on a set which is partitioned into sorts parameterized by the objects of $\mathbf{C}$ and such that whenever $C' \xrightarrow{\lambda} C$ is a morphism in $\mathbf{C}$ and x is an element of X of sort C, then $x\lambda$ is specified as an element of X of sort C', all this being subject to the conditions

$$\boxed{\begin{array}{l} x1_C = x, \\ x(\lambda\mu) = (x\lambda)\mu \quad \text{whenever} \quad C'' \xrightarrow{\mu} C' \xrightarrow{\lambda} C \quad \text{in} \quad \mathbf{C}. \end{array}}$$

Such an action X is also often referred to as a $\mathbf{C}$-set when there is little danger of confusion with other possible uses of that term. Similarly the $\mathbf{C}$-naturality of any morphism $X \xrightarrow{f} Y$ in the category $\mathcal{S}^{\mathbf{C}^{op}}$ is really just the homogeneity condition

$$f(x\lambda) = f(x)\lambda$$

wherein, of course, the first action of $\lambda \in \mathbf{C}$ is the one given by X and the second by Y. The Yoneda-Grothendieck-Dedekind-Cayley embedding

$$\mathbf{C} \xrightarrow[T_{\mathbf{C}}]{} \mathcal{S}^{\mathbf{C}^{op}}$$

is the functor which associates to each object A of C the $\mathbf{C}$-set $T_{\mathbf{C}}(A) = \mathbf{C}(-, A)$ whose C-th sort is the set $\mathbf{C}(C, A)$ of $\mathbf{C}$ morphisms $C \longrightarrow A$, with action by composition: $C' \xrightarrow{\lambda} C \xrightarrow{x} A$; this is a functor because for any $A \xrightarrow{\alpha} A'$ we get an induced $\mathbf{C}$-homogeneous map $\mathbf{C}(-A) \longrightarrow \mathbf{C}(-, A')$ which by the associativity of composition in $\mathbf{C}$ behaves functorially under composition $A \longrightarrow A' \longrightarrow A''$. The famous *lemma* of the four illustrious mathematicians says that $T_{\mathbf{C}}$ is full embedding $\mathbf{C} \hookrightarrow \mathcal{S}^{\mathbf{C}^{op}}$ (so that it is justified and often convenient to confuse A with $T_{\mathbf{C}}(A) = \mathbf{C}(-, A)$) and says much more, namely that for *any* $\mathbf{C}$-set X and for any object A of $\mathbf{C}$, the set of elements of X of sort A is naturally identifiable with the set of $\mathcal{S}^{\mathbf{C}^{op}}$-morphisms from

$\mathbf{C}(-, A)$ to X. It is thus justified, as well as extremely useful, to adjoin to the above parenthetically-introduced abuse still a further abuse of notation and to henceforth regard the elements of X of sort A as morphisms $A \longrightarrow X$ in $\mathcal{S}^{\mathbf{C}^{op}}$; thus the action of $\mathbf{C}$ on any $\mathbf{C}$-set becomes a special case of composition of morphisms

$$\begin{tikzcd} C' \arrow[r, "\lambda"] \arrow[dr, "x\lambda"'] & C \arrow[d, "x"] \\ & X \end{tikzcd}$$

now all in $\mathcal{S}^{\mathbf{C}^{op}}$ as does the application of a morphism f to an element, and the homogeneity (or naturality) property of every $X \longrightarrow Y$ in $\mathcal{S}^{\mathbf{C}^{op}}$ becomes a special case of the associativity of composition in $\mathcal{S}^{\mathbf{C}^{op}}$:

$$\begin{tikzcd} C' \arrow[r, "\lambda"] \arrow[dr, "x\lambda"'] & C \arrow[d, "x"] \arrow[dr, "fx"] & \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

In case the objects and morphisms of $\mathbf{C}$ have some kind of geometrical interpretation, it is often helpful to imagine that the more general objects of $\mathcal{S}^{\mathbf{C}^{op}}$ push that interpretation to a natural limit: an object A of $\mathbf{C}$ may be considered in $\mathcal{S}^{\mathbf{C}^{op}}$ as a generic "figure" and any $A \xrightarrow{x} X$ as a particular figure in X (quite possibly singular, i.e. not necessarily monomorphic) of sort A. Then if $x' = x\lambda$ in X one may consider that λ establishes change of figures in X and that an equation $x_1\lambda_1 = x_2\lambda_2$ is an incidence relation; the naturality or homogeneity of morphisms $X \longrightarrow Y$ is therefore essentially the preservation of incidence relations. One must distinguish between elements (figures of any generic sort) and points. By the latter we understand morphisms $1 \longrightarrow X$ where 1 is the terminal object of $\mathcal{S}^{\mathbf{C}^{op}}$, defined as the $\mathbf{C}$-set which for any A has exactly one element (figure) of sort $A : A \xrightarrow{A} 1$. Then it easily follows that for any X there is exactly one morphism $X \longrightarrow 1$, also denoted by X. [This useful abuse (due to Johnstone) is justified by the fact that for any object X of any category $\mathcal{X}$, the comma category $\mathcal{X}/X$ has a terminal object 1_X which is none other than the identity map of X in $\mathcal{X}$, and that for any object E of $\mathcal{X}/X$ the unique $E \longrightarrow 1_X$ is represented in $\mathcal{X}$ as ϵ

$$\begin{tikzcd} E \arrow[rr, "\epsilon"] \arrow[dr, "\epsilon"'] & & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

where ϵ is the structural datum in $\mathcal{X}$ that any object in $\mathcal{X}/X$ must have.]

Now a morphism $1 \xrightarrow{x} X$ must operate at each sort, picking an element x_A of X of each sort A; but since the action of $\mathbf{C}$ on 1 by any λ is trivial, and

x is homogeneous, we therefore have

$$x_{C'} = x_C\lambda \quad \text{for all} \quad C' \xrightarrow{\lambda} C \quad \text{in} \quad \mathbf{C}.$$

For example, if $1 \in \mathbf{C}$ [i.e. if $\mathbf{C}$ itself has a terminal object, in which case it is easy to see that $\mathbf{C}(-, 1) = 1$ the terminal object of $\mathcal{S}^{\mathbf{C}^{op}}$, a further justification for the Yoneda abuse] then any point x of X is determined by a single figure x_1 of sort 1, by $x_C = x_1C$ for all $C \xrightarrow{c} 1$ in $\mathbf{C}$. Since in combinatorial geometry there are often many higher-dimensional figures with few specified vertices, and since in dynamical systems there are often no rest states at all, it is not surprising that morphisms $X \longrightarrow Y$ are often not determined by their values on points alone: we may have

$$1 \overset{x}{\dashrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y \quad fx = gx \quad \text{for all such } x$$

$$f \neq g.$$

Of course if $f \neq g$ then there is some A in $\mathbf{C}$ and some figure x of sort A in X with $fx \neq gx$ in Y. It is sometimes interesting to consider the full subcategory $\mathcal{Y}_{\mathbf{C}}$ of $\mathcal{S}^{\mathbf{C}^{op}}$ consisting of all those Y such that for all A in $\mathbf{C}$ and any two figures $A \underset{y_2}{\overset{y_1}{\rightrightarrows}} Y$ with $y_1 \neq y_2$ there exists a point $1 \xrightarrow{a} A$ for which $y_1a \neq y_2a$; in other words, there is some point in the generic figure A such that the a-th *vertex* of y_1 differs from the a-th vertex of y_2. Consideration of $\mathcal{Y}_{\mathbf{C}}$ may seem especially pertinent when $1 \in \mathbf{C}$ and when $\mathbf{C} \subset \mathcal{Y}_{\mathbf{C}}$; however, we will *always* have $\mathcal{Y}_{\mathbf{C}} \neq \mathcal{S}^{\mathbf{C}^{op}}$ unless $\mathbf{C} \cong 1$. $\mathcal{Y}_{\mathbf{C}}$ is cartesian closed (see below) but does not have a truth-value object (see below) and is hence never a topos unless $\mathbf{C} \cong 1$; the advantage of being able to treat also the power objects etc. in $\mathcal{S}^{\mathbf{C}^{op}}$ as generalized $\mathbf{C}$-objects, as well as the use of superior set-like exactness properties toposes enjoy, derives from the fact that many conceptual constructions on objects (even if starting from objects of $\mathcal{Y}_{\mathbf{C}}$) will naturally lead to objects of $\mathcal{S}^{\mathbf{C}^{op}}$. Since $\mathcal{Y}_{\mathbf{C}}$ is closed under cartesian products Π and arbitrary subobjects in $\mathcal{S}^{\mathbf{C}^{op}}$, it follows that for every X in $\mathcal{S}^{\mathbf{C}^{op}}$ there is a natural surjective map $X \longrightarrow X^b$ to an object of $\mathcal{Y}_{\mathbf{C}}$ such that

$$\forall Y \in \mathcal{Y}_{\mathbf{C}} \forall f \exists! \overline{f} \qquad \begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & X^b \arrow[d, dotted, "\overline{f}"] \\ & Y \end{tikzcd}$$

The reader may wish to calculate X^b for some of the kinds of graphs discussed below.

The condition $1 \in \mathbf{C}$ may for some $\mathbf{C}$ not be strictly true yet quasi-true up to "splitting idempotents". There is for any $\mathbf{C}$ a "Cauchy-completion" $\mathbf{C} \subseteq \overline{\mathbf{C}} \subset \mathcal{S}^{\mathbf{C}^{op}}$ consisting of all retracts R of objects of $\mathbf{C}$ in the sense that there exist $R \underset{p}{\overset{i}{\rightleftarrows}} A$ morphisms of $\mathcal{S}^{\mathbf{C}^{op}}$ such that $A \in \mathbf{C}$, $pi = 1_R$. It is always

the case that there is an equivalence of categories $\mathcal{S}^{\overline{\mathbf{C}}^{op}} \xrightarrow{\sim} \mathcal{S}^{\mathbf{C}^{op}}$. By $1 \in \overline{\mathbf{C}}$, we thus mean that for some A in $\mathbf{C}$ there is for every C a morphism $C \xrightarrow{e_C} A$ such that $e_C\lambda = e_{C'}$ for *all* λ; thus in particular e_A is an idempotent but of the very special kind sometimes called a "right zero" (unlike two-sided zeroes, such are not necessarily unique, as we will see). In general the idempotents of $\mathbf{C}$ correspond to objects of $\overline{\mathbf{C}}$, and one says that $\mathbf{C}$ is "closed under splitting of idempotents" or "Cauchy-complete as an $\mathcal{S}$-category" if $\mathbf{C} \xrightarrow{\sim} \overline{\mathbf{C}}$ is an equivalence of categories. $\overline{\mathbf{C}}$ can be constructed purely abstractly from $\mathbf{C}$ without reference to actions by defining, for any $A_i \circlearrowleft_{e_i}$, $i = 1, 2$ with $e_i^2 = e_i$,

$$\overline{\mathbf{C}}(e_1, e_2) = \{\lambda \in \mathbf{C}(A_1, A_2) \mid \lambda e_1 = \lambda = e_2\lambda\}$$

noting the *two* equations, and verifying the required properties, including 2-functoriality and $\overline{\mathbf{C}} \xrightarrow{\sim} \overline{\mathbf{C}}$ for any $\mathbf{C}$. Note that $1_e \in \overline{\mathbf{C}}(e, e)$ is just e in $\mathbf{C}$. [For example, if $\mathbf{C}$ is the category of all smooth maps between all open subsets of all Euclidean spaces, then $\overline{\mathbf{C}}$ is the category of all smooth manifolds. This powerful theorem justifies bypassing the complicated considerations of charts, coordinate transformations, and atlases commonly offered as a "basic" definition of the concept of manifold. For example the 2-sphere, a manifold but not an open set of any Euclidean space, may be fully specified with its smooth structure by considering any open set A in 3-space E which contains it but not its center (taken to be 0) and the smooth idempotent endomap of A given by $e(x) = x/|x|$. All general constructions (i.e. functors into categories which are Cauchy complete) on manifolds now follow easily (without any need to check whether they are compatible with coverings, etc.) provided they are known on the opens of Euclidean spaces: for example, the tangent bundle of the sphere is obtained by splitting the idempotent e' on the tangent bundle $A \times V$ of A (V being the vector space of translations of E) which is obtained by differentiating e. The same for cohomology groups, etc.]

Even if $\mathbf{C}$ is a monoid, i.e. a category with one object C, $\overline{\mathbf{C}}$ will not be a monoid if $\mathbf{C}$ has idempotent elements other than the identity 1_C. It will often not even be equivalent to $\mathbf{C}$ [the question of equivalence is different since $\mathbf{C}$ could have elements f, g with $fg = 1_C$ but $gf \neq 1_C$ then $e \underset{\text{def}}{=} gf$ determines an object $e \neq 1_C$ in $\overline{\mathbf{C}}$, but $e \xrightarrow{f} 1_C$ is an isomorphism in $\overline{\mathbf{C}}$; in other words, *such* an idempotent e would already be split through C itself, but in a non-trivial way.]

Since we are primarily interested in $\mathcal{S}^{\mathbf{C}^{op}} \simeq \mathcal{S}^{\overline{\mathbf{C}}^{op}}$, we will freely pass back and forth between $\mathbf{C}$ and $\overline{\mathbf{C}}$ in describing objects X: for minimalistic sufficiency, we may prefer $\mathbf{C}$, especially when it is a monoid; but $\overline{\mathbf{C}}$ usually gives a more plastic vision of the kinds of figures which X really has and their relationships.

The most important example of the above, and probably of our whole discussion to follow, is the three element monoid Δ_1:

$$1_{\mathbf{I}}, e_0, e_1 \qquad e_i e_j = e_i \qquad i, j = 0, 1$$

This is a non-commutative monoid consisting entirely of idempotents (therefore called a "band" by semigroup theorists) but the other two equations tell us that moreover the two new objects in $\overline{\Delta}_1$ are both *terminal* objects (so we may say $1 \in \overline{\Delta}_1$) and hence in particular isomorphic. Thus we have

$$\begin{tikzcd} \Delta_1 \arrow[dr] & \subset & \overline{\Delta}_1 \arrow[d] \\ & & \boxed{1 \rightleftarrows \mathbf{I}} \end{tikzcd}$$

where the vertical functor is an equivalence of categories with two quasi-inverses and all three resulting presheaf toposes are essentially the same. In the two-object category pictured, the original monoid is recovered as all the endomorphisms of $\mathbf{I}$, and

$$e_i = \partial_i \mathbf{I} \qquad i = 0, 1$$

where $1 \xrightarrow{\partial_i} \mathbf{I} \xrightarrow{\mathbf{I}} 1$ is of course 1_1. We may even abbreviate ∂_i to i, in which case we can say that 0,1 are the only *points* of $\mathbf{I}$ even in $\mathcal{S}^{\Delta_1^{op}}$ but that $\mathbf{I}$ has one further element, namely the figure $1_{\mathbf{I}}$. In general a figure $\mathbf{I} \xrightarrow{X} X$ is often called an "edge" of X and the two points $x\partial_0$, $x\partial_1$ are the initial and terminal vertices of x

$$x_i = x\partial_i \qquad \begin{tikzcd} 1 \arrow[r, shift left, "\delta_0"] \arrow[r, shift right, "\delta_1"'] \arrow[dr, shift left, "x_0"'] \arrow[dr, shift right, "x_1"] & \mathbf{I} \arrow[d, "x"] \\ & X \end{tikzcd}$$

Given any point $1 \xrightarrow{p} X$ of X, there is a corresponding "degenerate edge" at p $\mathbf{I} \xrightarrow{\mathbf{I}} 1 \xrightarrow{p} X$ whose initial and terminal vertices are both p. In general an edge x for which $x\partial_0 = x\partial_1$ is called a "loop" at the corresponding point; there may be several loops at p but the degenerate loop $p\mathbf{I}$ is always among them though it may be the only one. Thus we have in this particular topos a good way of picturing the "inside" of any object X, the general elements as *arrows,* but the degenerate loops as *dots.* Thus for $\mathbf{I}$ itself we have a single non-degenerate arrow:

$$\mathbf{I} = \boxed{\underset{0}{\bullet} \xrightarrow{1_{\mathbf{I}}} \underset{1}{\bullet}}$$

and there are morphisms

$$\begin{tikzcd} \mathbf{I} \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, equal] & 1 \arrow[d, equal] \\ \boxed{\bullet \to \bullet} \arrow[r] & \boxed{\circlearrowleft} \arrow[r] & \boxed{\bullet} \end{tikzcd}$$

This illustrates that

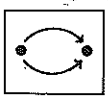
1) [in *contrast* to the category $\mathcal{S}^{\mathbf{P}^{op}}$ of "irreflexive" graphs where there are not necessarily any loops and where even if there are loops there is no specific one preserved by morphisms: $\mathbf{P} = \boxed{\bullet \rightrightarrows \bullet}$ is a *category* which happens

to be determined by its underlying reflexive graph structure] in our "reflexive" graphs $\mathcal{S}^{\Delta_1^{op}}$ an edge may become degenerate under the application of a morphism such as $C \to 1$ or $\mathbf{I} \to 1$, and

2) it may be useful to consider quotients (like C) of objects of $\mathbf{C}$ (even when $C \notin \overline{\mathbf{C}}$) as further "generic" figures, so that various types of singular figures (such as loops) also become "representable" at least by objects of $\mathcal{S}^{\mathbf{C}^{op}}$: the surjection $\mathbf{I} \twoheadrightarrow C$ induces the inclusion

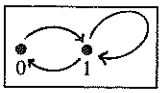
$$(C, X) \hookrightarrow (\mathbf{I}, X)$$

of the set of all loops into the set of all edges for any reflexive graph X. Another important figure type in this category is

$E =$ 

Note that there is a morphism $E \xrightarrow{p} \mathbf{I}$ which has *two* sections $\mathbf{I} \underset{s'}{\overset{s}{\rightrightarrows}} E$ (i.e. $ps = 1_{\mathbf{I}} = ps'$) but that there are two more morphisms $\mathbf{I} \longrightarrow E$ which are not sections of p (in fact they factor across 1) because E has four edges in all. E is not even a quotient of any object of $\overline{\mathbf{C}}$, but it is a quotient of $2\mathbf{I} = \mathbf{I} + \mathbf{I}$ the disjoint sum (coproduct) of two copies of $\mathbf{I}$.

An object of $\mathcal{S}^{\Delta_1^{op}}$ which has several universal significances is

$\Omega =$ 

which has two points but five edges in the configuration shown (which uniquely specifies the action of ∂_0, ∂_1 on all five). The most basic of these is that it classifies subgraphs. Here by a subgraph of X we mean (not a graph with a property but) a graph A equipped with a specified "inclusion" morphism $A \longrightarrow X$, denoted for example by i_A when it is not understood: for example there are two *different* subobjects of $\mathbf{I}$ which, without their inclusions, are the same 1, and two *different* subobjects of E which without their inclusions are the same $\mathbf{I}$. The only condition for a morphism i (in a topos such as our presheaf toposes; one might well need to complicate this in a non-topos such as the $\mathcal{Y}_{\mathbf{C}}$ mentioned before) to be a subobject inclusion is that it be a *mono*morphism, i.e. satisfy the cancellation property $ia_1 = ia_2 \Rightarrow a_1 = a_2$ for any $T \underset{a_2}{\overset{a_1}{\rightrightarrows}} A$; this $\mathbf{I}$ has essentially five subobjects $0, \mathbf{I}$, $0, 1$, $\{0, 1\}$ where the last is $2 \hookrightarrow \mathbf{I}$ with $2 = 1 + 1$ the "discrete" graph, while E has seven and for the generic loop C we get three. Here "essentially" refers to isomorphism in the comma category $\mathcal{S}^{\Delta_1^{op}}/X$ with $X = \mathbf{I}$ or $X = E$ or $X = C$ for example. In more detail, we write $A \underset{X}{\subseteq} B$ to mean we are given i_A, i_B monomorphisms and can find β so that $i_A = \beta i_B$ in

$$\begin{tikzcd} A \arrow[rr, dashed, "\beta"] \arrow[dr, "i_A"'] & & B \arrow[dl, "i_B"] \\ & X & \end{tikzcd}$$

We also write $x \in A$ to mean that x is any morphism with codomain X while we are given a *mono*morphism i_A and can find an α such that $x = i_A\alpha$

$$\begin{tikzcd} T \arrow[rr, dashed, "\alpha"] \arrow[dr, "x"'] & & A \arrow[dl, "i_A"] \\ & X & \end{tikzcd}$$

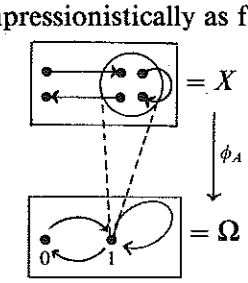
In either case the β or the α which "proves" the inclusion or membership is unique because the second map being compared is monic; in the first case β is also monic since more generally $k\beta$ monic implies β is even if k isn't. The fact that inclusion is a special case of membership, and in the case of points membership a special case of inclusion, of course plays havoc with mid-century axiomatic set-theory but accords well with the naive set theory and geometry which has survived since well before that. For example, we obviously have (by composition in the comma category of "proofs")

$$x \in A \;\&\; A \underset{X}{\subset} B \quad \Rightarrow x \in B$$

and the resulting quantified implication

$$A \underset{X}{\subset} B \Rightarrow \forall x[x \in A \Rightarrow x \in B]$$

can be *reversed* (trivially since we can take $x = i_A$); it can (less trivially) still be reversed in a category like $\mathcal{S}^{\mathbf{C}^{op}}$ even if we restrict the universal quantifier in the hypothesis to range only over those x whose domains $T \in \mathbf{C}$. Thus in particular a subgraph of X is determined by some of its edges and some of its points, with the only constraint that both vertices of any selected edge must be selected points. Now the claim that the five-edge graph Ω classifies subgraphs means that Ω has a subgraph $1 \xrightarrow{\text{true}} \Omega$ such that for any graph X and any subgraph $A \xrightarrow{i_A} X$ there is a unique morphism $X \xrightarrow{\varphi_A} \Omega$ such that i_A is the inverse of true under φ_A. That is, for all $T \xrightarrow{x} X$, $x \in A$ iff $\varphi x = \text{true}T$, and the "characteristic" map φ_A of i_A is the *only* morphism with this property. This uniquely determines Ω up to isomorphism. In the case $\mathcal{S}^{\Delta_1^{op}}$ of reflexive graphs, Ω is forced to be as claimed, with true the point at which the non-degenerate loop also resides, for the following reason. Picture the inclusion of A in X impressionistically as follows:



Of course all edges and points in A go by φ to the one true point, and all *points* of X not in A must go to the other place "false"; but there may be

edges of X which initiate or terminate in A but not conversely and they will have to go to the appropriate arrow joining true and false since φ must be homogeneous with respect to ∂_0, ∂_1; there may well be paths between points of A which are provided by X but not by A, and they are forced to go by φ to the non-trivial loop at true. Thus all of Ω is needed because of the variety of inclusions that exist in $\mathcal{S}^{\Delta_1^{op}}$. On the other hand, taking anything bigger than Ω would ruin the *uniqueness* of φ. Collapsing Ω, for example $\Omega \longrightarrow \Omega^b$ the reflection into the category $\mathcal{Y}_{\Delta_1}$ of graphs with no multiple edges, would induce a closure operation on subgraphs, but could only *classify* certain ones, in the case of Ω^b the "full" subgraphs.

Another "universal" property which happens to be true and interesting in the $\mathcal{S}^{\Delta_1^{op}}$ province is that there are enough morphisms $X \longrightarrow \Omega$ to distinguish edges, so that if $F(X)$ denotes the abstract set $F(X) = (X, \Omega)$ of subgraphs of X and $\Omega^{F(X)}$ the $F(X)$-fold cartesian product of copies of Ω, then the canonical morphism

$$\begin{aligned} X &\longrightarrow \Omega^{F(X)} \\ x &\mapsto [\varphi \longrightarrow \varphi x] \end{aligned}$$

is actually a *subgraph* ! (of course in a particular case a much smaller subset $S \subset F(X)$ may suffice).

In any category of the form $\mathcal{S}^{\mathbf{C}^{op}}$ there is such a distinguished truth-value object $\Omega_{\mathbf{C}}$, whose elements of sort C "are" just all the ($\mathcal{S}^{\mathbf{C}^{op}}/C$-isomorphism types of) subactions of C; such is just any set of arrows $B \longrightarrow C$ for various B in $\mathbf{C}$, subject only to the requirement that for any $B' \longrightarrow B$ in $\mathbf{C}$, and $B \longrightarrow C$ in the set, we must also have the composite $B' \longrightarrow C$ in the set, in other words, any "right ideal" of $\mathbf{C}$ in the case of a monoid. The action of $\mathbf{C}$ on $\Omega_{\mathbf{C}}$ is by inverse image (or "division" or "analysis"): If S is a right ideal in C and if $C' \xrightarrow{\lambda} C$, then

$$t \in S\lambda \quad \text{iff} \quad \lambda t \in S;$$

the requisite property $S(\lambda\mu) = (S\lambda)\mu$ for $C'' \xrightarrow{\mu} C' \xrightarrow{\lambda} C$ then follows. The object Ω thus *constructed* to classify subactions of the $\mathbf{C}(-, C)$ then actually succeeds to classify subactions of any X, for given a subobject A of X, we can define φ by

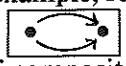
$$\boxed{t \in \varphi x \quad \text{iff} \quad xt \in A}$$

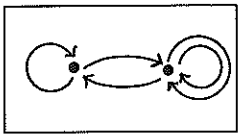
$$\begin{tikzcd}
B \arrow[dr, "t"'] \arrow[drr, dashed] & & & \\
& C \arrow[drr, "x"'] & A \arrow[r] \arrow[d, "i_A"] & 1 \arrow[d, "\text{true}"] \\
& & X \arrow[r, "\varphi"'] & \Omega
\end{tikzcd}$$

(where of course true$C = \mathbf{C}(-, C)$ the greatest of all right ideals in C). In other words, the truth-value φx of the statement "$x \in A$" is identified engineering-wise with the set of all possible acts t which would bring about its actual truth. [This suggests a notion of numerical "measures" on the $\cup$-lattice Ω, whose theory is so far only fragmentarily developed.]

The functor $\mathcal{S}^{\mathbf{C}^{op}} \xrightarrow{(1,-)} \mathcal{S}$ assigning to every $\mathbf{C}$-set its abstract set of *points* always has a left adjoint assigning to each set the corresponding *discrete* $\mathbf{C}$-set which has S elements of each sort and in which $s_{C'} = s_C\lambda$ for all λ for each $s \in S$, and each $C' \xrightarrow{\lambda} C$ in $\mathbf{C}$. Composing these two adjoints we get (in the case of graphs) the maximal discrete subgraph $|X| \hookrightarrow X$ of any graph, whose characteristic map $X \xrightarrow{\delta} \Omega$ factors through the loop $C \hookrightarrow \Omega$. For any morphism $X \xrightarrow{f} Y$, the composite $\delta_Y f$ classifies the subgraph D_f of X which is the *degeneracy* of f.

On the other hand the "points" functor $\mathcal{S}^{\mathbf{C}^{op}} \xrightarrow{(1,-)} \mathcal{S}$ will have a right adjoint iff $1 \in \overline{\mathbf{C}}$; this assigns to any set S the codiscrete or chaotic action whose figures of C-th sort are the elements of $S^{(1,C)}$. Thus for $\mathbf{C} = \Delta_1$, the chaotic graph on S points has S^2 edges.

To pass to another example, recall that we remarked before the elementary "parallel process" $E =$  is a reflexive graph which happens to admit only one definition of composition making it into a category $\mathbf{P}$. This also happens to be a self-dual category in the sense that there exists an isomorphism $\mathbf{P}^{op} \xrightarrow{\sim} \mathbf{P}$ of categories. Its actions $\mathcal{S}^{\mathbf{P}^{op}}$ are the *irreflexive* graphs (the negative is in a way appropriate even for those objects which happen to have loops at some point p, for morphisms are allowed to interchange *any* two such loops). It has no sense here to speak of degenerating via a morphism since $\mathbf{P} = \boxed{U \rightrightarrows \mathbf{I}} \hookrightarrow \mathcal{S}^{\mathbf{P}^{op}}$ where $0 \subseteq U \subseteq 1$. Indeed the "points" $1 \longrightarrow X$ should really be pictured here as LOOPS and the "nodes" $U \longrightarrow X$ pictured as dots. Our *new* $\mathbf{I}$ still has five subobjects in $\mathcal{S}^{\mathbf{P}^{op}}$ but $\Omega = \Omega_{\mathbf{P}}$ must be pictured as

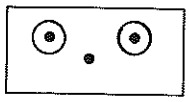


There are three morphisms $1 \longrightarrow \Omega$ (i.e. loops in Ω) since the terminal object 1 itself (the generic loop now) has three subobjects. We can choose either of the two loops at the one node as "true"; the node itself can't be so chosen since in any topos it can be shown that the generic subobject $G \hookrightarrow \Omega$ must have a domain $G = 1$ (i.e. in this case a loop) and not for example $U \subseteq 1$. The unique non-trivial automorphism of Ω induces an involutory modal "negation" operator on subobjects of any object X in $\mathcal{S}^{\mathbf{P}^{op}}$, different from the intuitionistic negation (which is also present) since it preserves false! Since $\mathbf{P}$ has cancellation, the comma category $\mathcal{S}^{\mathbf{P}^{op}}/\mathbf{I} \approx \mathcal{S}^{(\mathbf{P}/\mathbf{I})^{op}}$ is actually the

actions of the *poset*

$$\boxed{\begin{array}{l} U_0 \longrightarrow \mathbf{I} \\ U_1 \nearrow \end{array}}$$

which can usefully be considered as the basic open sets of a three-point topological space



in which one point is not open, making five open sets in all. The "quotient map" $\mathcal{S}^{\mathbf{P}^{op}}/\mathbf{I} \longrightarrow \mathcal{S}^{\mathbf{P}^{op}}$ identifies the two basic regions without thereby coalescing the actions of the corresponding restriction maps from the whole. Note that *discrete* $\mathbf{P}$ sets are just disjoint sums of single loops, so that the analogue (pullback) of the degeneracy D_f of a map $X \xrightarrow{f} Y$ is not a sub-object of X; the fibers of $|Y| \longrightarrow Y$ disconnect the multiple loops, and nodes which have no loops are not covered at all.

Our three-element example $\mathbf{\Delta}_1$ could even more concretely be realized as the full subcategory of Posets or of Cat consisting of (1 and) 2 in which case we have the adjointness relations

$$\partial_0 \dashv \mathbf{I} \dashv \partial_1$$

uniquely determining all by any one, and hence for any category $\mathbf{A}$ there is a reflexive graph *in Cat* (rather than in $\mathbf{S}$) in which the graph structure is determined by adjointness $\mathbf{A}^2 \rightleftarrows \mathbf{A}$. A more general kind of such "adjoint graph" is

$$\mathcal{S}^{\mathbf{C}^{op}} \rightleftarrows \mathcal{S}$$

in which the initial set of any "edge" action X is its set of (rest) points, the degenerate "edge" at any "point" $S \in \mathcal{S}$ is the trivial action, and the terminal set of any X is its set of components. In an example like $\mathbf{C} = \mathbf{\Delta}_1$, every component contains a point, and the equivalence condition on pairs of points of $|X|$ induced by the surjection

$$\begin{tikzcd} |X| \arrow[r] \arrow[dr] & X \arrow[d] \\ & \pi_0 X \end{tikzcd}$$

is also the one generated by the existence of two actors $\lambda \in \mathbf{C}$ taking one element to the two points. Thus it is apparent from its picture that $\pi_0\Omega = 1$.

As another example we might consider the four element monoid $\mathbf{F}$ of all endomaps of a two element set. The two inclusion functors $\mathbf{\Delta}_1 \leftrightarrows \mathbf{F}$ show that any $\mathbf{F}$-set X may be viewed as a directed graph in two ways, with the *same* set of *points*. The additional act τ with $\tau^2 = 1_{\mathbf{I}}$ in $\mathbf{F}$ *reverses* each edge of X to a canonically associated edge of X going the opposite way, and this canonical return trip is preserved by morphisms $X \longrightarrow Y$. There are

objects in which some non degenerate loops are fixed by τ, for example the two element C'. This category $\mathcal{S}^{\mathbf{F}^{op}}$ has been extensively used to proving nontrivial theorems about free groups on the basis of geometric intuition by Serre, Bass, Gersten, Duskin, and others, though without ever yet exploiting its topos structure! The connection is as follows: Since $\mathbf{F} \hookrightarrow \mathbf{S}$ is a full subcategory of the category $\mathbf{S}$ of non empty finite sets, we get a full inclusion $\mathcal{S}^{\mathbf{F}^{op}} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}}$ by taking the right adjoint of the restriction (as well as another full inclusion by taking the left adjoint; it is the left adjoint which extends $\mathbf{F} \hookrightarrow \mathbf{S}$ with respect to the Yoneda embeddings $T_{\mathbf{F}}$, $T_{\mathbf{S}}$, but the right adjoint which is considered the more basic inclusion in sheaf theory). $\mathcal{Y}_{\mathbf{S}} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}}$ is the classical category of simplicial complexes, which explains the relation to combinatorial topology. A third reflective subcategory of $\mathcal{S}^{\mathbf{S}^{op}}$ is the category $\mathcal{G}$ of groupoids, i.e. categories in which every morphism is an isomorphism, and functors (= homomorphisms) between these; since $1 \in \mathbf{S}$, there is a *codiscrete* (chaotic) inclusion $\mathcal{S} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}}$ right adjoint to "points" = (1,-) which takes any set S to the action whose elements of sort n are just S^n, for $n \in \mathbf{S}$ and where $n' \xrightarrow{\lambda} n$ in $\mathbf{S}$ acts by composing $n' \xrightarrow{\lambda} n \xrightarrow{x} S$. Now if we consider elements of $(\underline{2}, X)$ as arrows between the points $(1, X)$, X already suggests how to compose these via the elements a of $(\underline{3}, X)$, whose "boundary" elements are deduced via inclusions $2 \hookrightarrow 3$ in $\mathbf{S}$ and might be pictured as

$$\begin{tikzcd} & 1 \arrow[dr, "x_0"] \arrow[d, phantom, "a"] & \\ 0 \arrow[ur, "x_2"] \arrow[rr, "x_1"'] & {} & 2 \end{tikzcd}$$

We consider "$a \models x_1 \equiv x_0 x_2$" as a relation to be imposed on the free groupoid generated by words from $(\underline{2}, X)$, for each $a \in (\underline{3}, X)$, thus obtaining the groupoid $\pi_1 X$ and the left adjoint

$$\mathcal{S}^{\mathbf{S}^{op}} \xrightarrow{\pi_1} \mathcal{G}$$

to the full inclusion which to each groupoid $\mathbf{G}$ associates the composable strings of arrows from $\mathbf{G}$. The latter is acted on by *all maps* $n' \xrightarrow{\lambda} n$ since where λ is not surjective, we can compose in $\mathbf{G}$ some segments of an n-string, where it is not injective we can insert some identities into the string to bring it up to length n', and where symbols are "interchanged" we can imagine using the $(\ )^{-1}$ operation of the groupoid. A more unified description of the inclusion is obtained if we note that there is also a codiscrete inclusion $\mathbf{S} \hookrightarrow \mathcal{G}$, since if $\underline{S}(i,j) \underset{\text{def}}{=} 1$ for all $\langle i, j\rangle$ in S^2, there is a unique way to compose these, and $\underline{S}$ is a groupoid with S objects since $\underline{S}(i,j) \xrightarrow{(\ )^{-1}} \underline{S}(j,i)$ just reverses. This restricts to $\mathbf{S} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{G}$, which then induces $\mathcal{G} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}}$ immediately by $(\underline{n}, \mathbf{G}) \underset{\text{def}}{=} \mathcal{G}(\underline{n}, \mathbf{G})$; this is full since any homogeneous map $X \xrightarrow{f} Y$ between *such* actions of $\mathbf{S}$ preserves $(\underline{3}, X) \longrightarrow (\underline{3}, Y)$ which are

essentially the multiplication tables of the original groupoids, hence f comes from a homomorphism between the latter. The composites $\mathcal{S}^{\mathbf{F}^{op}} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}} \longrightarrow \mathcal{G}$ and $\mathcal{Y}_{\mathbf{S}} \hookrightarrow \mathcal{S}^{\mathbf{S}^{op}} \longrightarrow \mathcal{G}$ thus provide Poincaré groupoids for $\mathbf{F}$-graphs and simplicial schemes respectively. Lemmas showing how to spread out the graphs thus giving rise to free groups lead to geometric proofs of theorems of Schreier, Gersten, etc.

Besides $\Omega_{\mathbf{C}}$, the other crucially topos-theoretic property of the presheaf categories $\mathcal{S}^{\mathbf{C}^{op}}$ is "cartesian closedness", that is the existence of exponential functors $(\ \ )^A$ characterized by their ("λ-conversion") right adjointness to the cartesian product functor $A \times (\ \ )$ for each $A \in \mathcal{S}^{\mathbf{C}^{op}}$:

$$\frac{X \longrightarrow Y^A}{A \times X \longrightarrow Y}$$

The object Y^A can be constructed using this adjointness applied to the special case $X = C \in \mathbf{C}$ by invoking Yoneda's lemma: the elements of the sort C in Y^A "are" just the arbitrary morphisms $A \times C \longrightarrow Y$, acted on by

$$A \times C' \xrightarrow{1_A \times \lambda} A \times C \longrightarrow Y \quad \text{for} \quad C' \xrightarrow{\lambda} C.$$

[Somewhat more conceptually, these elements are the morphisms $C^*A \to C^*Y$ in $\mathcal{S}^{\mathbf{C}^{op}}/C$ where C^*X denotes the trivial fibration $X \times C \to C$.] In the case of a monoid $\mathbf{C}$ with its single dominant sort $\mathbf{I}$, the objects of the form $Y^{\mathbf{I}}$ (e.g. for $Y = \mathbf{I}$ itself, or $Y = \Omega_{\mathbf{C}}$) present themselves as the simplest interesting cases for calculation: $Y^{\mathbf{I}}$ has as elements the maps $\mathbf{I} \times \mathbf{I} \xrightarrow{f} Y$ satisfying

$$f(x\lambda, t\lambda) = f(x,t)\lambda$$

remembering that the elements of $\mathbf{I}$ are the elements of the original monoid itself. The action of $\mathbf{C}$ on this set of maps is

$$(f\lambda)(x,t) = f(x, t\lambda)$$

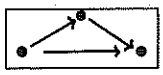
by acting only on the "test" component t, not on the one that is to be exponentiated. There is as yet no very systematic way of calculating even the case $\mathbf{I}^{\mathbf{I}}$, which therefore must be done one $\mathbf{C}$ at a time. For example, if $\mathbf{C}$ is the monoid of all continuous self maps of the unit interval $[0, 1]$, so that the points $1 \longrightarrow \mathbf{I}$ in $\mathcal{S}^{\mathbf{C}^{op}}$ are the points of $[0, 1]$ in the usual sense, then the maps $\mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ are just the usual continuous functions of two variables, which is mildly surprising, but not too difficult to prove. The corresponding statement for the monoid of *smooth* ($= C^\infty$) self maps of the line is surprising (once one realizes that one has to show that all the higher formally defined partial derivatives are the actual partial derivatives so in particular commute) and rather difficult to prove (Bowman 1966 and forthcoming book by Frölicher and Kriegl). Having calculated $Y^{\mathbf{I}}$, even more interest attaches to the natural path functionals $Y^{\mathbf{I}} \longrightarrow Y$. In both of the examples mentioned, the real line R determines an object of $\mathcal{S}^{\mathbf{C}^{op}}$ with just the reals as points by defining its elements of sort $\mathbf{I}$ to be just the continuous (resp. smooth) paths in R. Since

multiplication is continuous it gives rise to a morphism $R \times R \longrightarrow R$ in $\mathcal{S}^{\mathbf{C}^{op}}$ and hence to a left action $R \times R^{\mathbf{I}} \longrightarrow R^{\mathbf{I}}$ by $(af)(x) = a(f(x))$. Thus (in the smooth case) one can look for the object of *linear* functionals

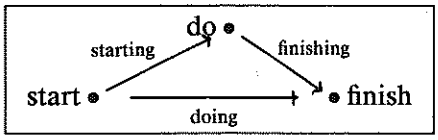
$$\mathrm{Lin}_R(R^R, R) \hookrightarrow R^{(R^R)}$$

whose points are just the morphisms $R^R \xrightarrow{\varphi} R$ which moreover satisfy $\varphi(af) = a\varphi(f)$. Schanuel, Zame, and I showed in 1980 that these are just the Schwartz distributions (of compact support); see the forthcoming book by Frölicher and Kriegl for details.

In the case $\mathbf{C} = \Delta_1$, the reader should be able to compute that the elements of $\mathbf{I}^{\mathbf{I}}$ are the six maps $\mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ which are the two constants, the two projections, and the maps corresponding to "max" and "min" (when we consider that $0 \leq 1$ in $\mathbf{I}$) but that the action of Δ_1 on this set is such that its standard picture as a graph is

$\mathbf{I}^{\mathbf{I}} =$ 

In an interpretation where $\mathbf{I} \longrightarrow X$ are thought of as processes, the following names for the elements of $\mathbf{I}^{\mathbf{I}}$ are suggestive:

$\mathbf{I}^{\mathbf{I}} =$ 

The reader is invited to correlate these six names with the names 0, 1, proj_1, proj_2, max, min. Computation of $\mathbf{I}^{\mathbf{I}} \longrightarrow \mathbf{I}$, $\Omega^{\mathbf{I}}$, etc. is further invited.

For presheaf categories $\mathcal{S}^{\mathbf{C}^{op}}$ there is the extremely important formula

$$\mathcal{S}^{\mathbf{C}^{op}}/X \cong \mathcal{S}^{(\mathbf{C}/X)^{op}}$$

showing that all the "slice" or "comma" categories (relative to any $\mathbf{C}$-set X) are again presheaf categories. Here $\mathbf{C}/X$ is the category whose objects are the figures of X and whose morphisms are the incidence relations: $x' \xrightarrow{\lambda} x$ where $x' = x\lambda$, $C' \xrightarrow{\lambda} C$ in $\mathbf{C}$, and x (resp. x') is a figure in X of sort C (resp. C'). The forgetful functor $\mathbf{C}/X \longrightarrow \mathbf{C}$ is what is called a (discrete op-) *fibration.*

For one example of the kind of "cohesiveness" which might be expressible by a directed graph X in $\mathcal{S}^{\Delta_1^{op}}$, consider a paragraph in which there are several concrete or abstract things talked about (and which are taken as points) whereas each sentence (edge) has a subject and an object (represented by the operations ∂_0, ∂_1). Many intransitive verbs can be considered as reflexive versions (loops) of transitive verbs, so that these too can be included in such an analysis. Since the paragraph contains many sentences, which make many interlocking statements about the things, a non-trivial graph structure thus results. (The degenerate loop at each point b may be considered as the sentence "b is b", which is implicit in the paragraph.) A translation or

interpretation of one paragraph into another (perhaps in another language) should at least be a morphism of graphs. But it should preserve more than just the "subject of a sentence" and "object of a sentence" incidence relations, and this can be partly expressed by passing to a category of the form $\mathcal{S}^{\Delta_1^{op}}/V$ where V is a fixed graph of "labels" or "values". Frequently V is a graph that consists entirely of loops; then for $X \to V$ to be a graph morphism merely means that every *degenerate* loop in X is mapped to a *point* of V. Thus for example V could be a classification of verbs such as that into "state, activity, achievement" verbs, including the single point "is", and a paragraph X could be given the structure of an $\mathcal{S}^{\Delta_1^{op}}/V$ object by mapping each sentence to the type of its principal verb (it might be reasonable to map some *non*-degenerate sentences such as "x_1 resembles x_2" into the point "is"); or tenses might also be included in V. Then a translation $X \longrightarrow Y$ would be required to preserve the labeling of X and Y

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & V & \end{tikzcd}$$

that is, to be a morphism in $\mathcal{S}^{\Delta_1^{op}}/V$.

The special role of labeling graphs as objects in $\mathcal{S}^{\Delta_1^{op}}/V$ where V has only loops comes up very frequently. For example, if the edges in X are to be interpreted as processes or as roads between towns and the labeling signifies time or cost or distance, the only *general* requirement is that a trivial process costs nothing, so V could be taken as a set of vectors or real numbers or other abstract quantities, all construed as loops at a single point, with the quantity 0 identified as the degenerate loop.

This special role of loops can be formalized as follows: There is a unique surjective homomorphism of monoids $\Delta_1 \longrightarrow \{0, 1\}$ from our three-element monoid to the multiplicative monoid consisting of the numbers 0, 1. (This is what results if we "force Δ_1 to become commutative".) The only significant feature of the element 0 is that it is a generic idempotent, so the objects of the category $\mathcal{S}^{\{0,1\}^{op}}$ may be identified with diagrams of sets

$$X_0 \underset{i}{\overset{p}{\leftrightarrows}} X, \qquad pi = id_{X_0}$$

the action of 0 being the composite ip on X (one of the great many kinds of examples of such a structure arises when we are given an arbitrary mapping $X_0 \xrightarrow{\alpha} X_1$, $X \underset{\text{def}}{=} X_0 \times X_1$, and i is taken to be the inclusion of the "graph" of α). In this case the discrete inclusion $\mathcal{S} \hookrightarrow \mathcal{S}^{\{0,1\}^{op}}$ not only has a right adjoint "points" functor and a left adjoint "components" functor, but these two functors are *isomorphic,* viz. to $X \mapsto X_0$; hence the components functor trivially preserves products (indeed all limits) and there is trivially a notion of "codiscrete" (which indeed coincides with "discrete"). However, the truth-value object $\Omega_{\{0,1\}}$ is *not* connected since it has three elements and *two* components. We will return to this remarkable topos (which in particular seems to be neither "gros" nor "petit") in the third section, but for

the moment let us return to its relationships with the topos $\mathcal{S}^{\Delta_1^{op}}$ of reflexive graphs.

The homomorphism $\Delta_1 \xrightarrow{q} \{0,1\}$ induces (as does any functor) three adjoint functors between presheaf categories

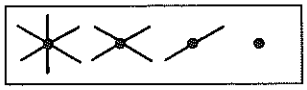
$$q_! \dashv q^* \dashv q_* \qquad \begin{tikzcd} \mathcal{S}^{\Delta_1^{op}} \arrow[d, shift right=3, "q_!"'] \arrow[d, shift left=3, "q_*"] \\ \mathcal{S}^{\{0,1\}^{op}} \arrow[u, "q^*"] \end{tikzcd}$$

where q^* simply reinterprets $X_0 \underset{i}{\overset{p}{\leftrightarrows}} X$ as a graph with X_0 points and X loops, the location of the loops being specified by p. This is the basic relationship required for the kind of "labeling" applications mentioned above, and in fact suggests considering the comma category (or "glued" topos) $\mathcal{S}^{\Delta_1^{op}}/q^*$ as a category of labeled graphs in which morphisms include the possibility of re-labeling. The right adjoint functor q_* essentially discards all the non-loops in an arbitrary graph, whereas the left adjoint $q!$ forces all edges to become loops by replacing the old set of points with the new set of points which is the set of components of the original graph; the new set of edges is a quotient set of the old in that all the *degenerate* loops in each given component become identified.

How should one internally picture the objects $X_0 \underset{i}{\overset{p}{\leftrightarrows}} X$, $pi = id_{X_0}$ of $\mathcal{S}^{\{(0,1\}^{op}}$? Probably there is no single preferred way, since objects so abstract have many diverse applications; however, one way suggested (by Meloni) is that the generic figure is

$\mathbf{I}_{\{0,1\}} =$ 

so that in general every object of $\mathcal{S}^{\{0,1\}^{op}}$ is a sum



of connected components (the same number as the number of points !) while each component (which is essentially nothing but a pointed set) is indicated by drawing one small path (without endpoints) through the point for each non degenerate element. Then the effect of the functor q^* is to close up each little path into a loop. There are more connections between the two toposes, induced by the two obvious injective homomorphisms

$$\{0,1\} \underset{i_1}{\overset{i_0}{\rightrightarrows}} \Delta_1$$

where $i_K(0) = e_K$ (recall that 0 is a generic idempotent; of course i_0, i_1 are both sections of q). Each of i_0, i_1 induce three adjoint functors as q does; for example i_0^* redefines every edge in a graph as a loop at its beginning point.

Although $\mathcal{S}^{\Delta_1^{op}}$ is just one topos and the toposes $\mathcal{S}^{\Delta_1^{op}}/B$ are still quite special, it can be shown that for any Grothendieck topos $\mathcal{X}$ there exists a graph B and an idempotent left-exact endofunctor E of $\mathcal{S}^{\Delta_1^{op}}/B$ such that $\mathcal{X}$ is equivalent to the subcategory of $\mathcal{S}^{\Delta_1^{op}}/B$ consisting of the objects fixed by E. As an example, consider the topos $\mathcal{S}^{\mathcal{F}(B)^{op}}$ of right actions of the free category (or path category) determined by a reflexive graph B. This is equivalent to the subcategory of $\mathcal{S}^{\Delta_1^{op}}/B$ consisting of all $X \longrightarrow B$ which satisfy the condition of "fibration" type:

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[d, "\partial_1"'] & X \arrow[d] \\ \mathbf{I} \arrow[r, "\beta"] \arrow[ur, dashed, "\overline{x}"] & B \end{tikzcd}$$

i.e. that for any edge β of B and any point x projecting to $\beta\partial_1$ there is a unique edge $\overline{x}$ which projects to β and has $\overline{x}\partial_1 = x$ (then $x \cdot \beta \underset{\text{def}}{=} \overline{x}\partial_0$ uniquely defines the "action" of β); and this implies that any edge $\overline{x}$ which projects to a degenerate b is itself already degenerate (then in particular we have a further derived "fibration" type condition

$$\begin{tikzcd} \mathbf{I} \arrow[r, "x"] \arrow[d] & X \arrow[d] \\ 1 \arrow[r, "b"] \arrow[ur, dashed] & B \end{tikzcd}$$

and each degenerate loop b *acts* as the identity). Under the equivalence

$$\mathcal{S}^{\Delta_1^{op}}/B \cong \mathcal{S}^{(\overline{\Delta}_1/B)^{op}}$$

this means that those morphisms in Δ_1/B which are over either $1 \xrightarrow{\partial_1} \mathbf{I}$ or $\mathbf{I} \longrightarrow 1$ in $\overline{\Delta}_1$ are being required to act as *bijections* between the fibers of X, or again that what is really acting on X is the category of fractions

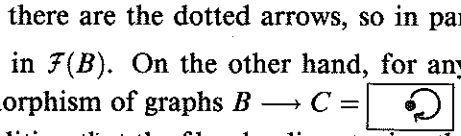
$$\overline{\Delta}_1/B \longrightarrow \mathcal{F}(B)$$

obtained by formally inverting $\mathcal{D}(B)$, the inverse image under $\overline{\Delta}_1/B \longrightarrow \overline{\Delta}_1$ of the subcategory $\mathcal{D}(1) = 1 \underset{I}{\overset{\partial_1}{\rightleftarrows}} \mathbf{I} \circlearrowleft_{e_1}$ of $\overline{\Delta}_1$. Every edge β in B determines a morphism in $\mathcal{F}(B)$, namely the one coming from

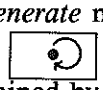
$$\sigma(\beta) = \boxed{\begin{tikzcd} 1 \arrow[rr, dash, "\partial_0"] \arrow[dr, dash, "b'"'] & & \mathbf{I} \arrow[dl, dash, "\beta"] \\ & B & \end{tikzcd}}$$

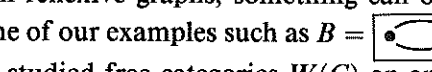
in Δ_1/B. But composites of these (hence longer words) can be formed in $\mathcal{F}(B)$, since from $b'' \xrightarrow{\alpha} b' \xrightarrow{\beta} b$ in B we get in $\overline{\Delta}_1/B$ the solid arrows

$$\begin{tikzcd}
 & & b \arrow[d, "\tau(\beta)"] \\
 & b' \arrow[ur, dashed, "\overline{\beta}"] \arrow[r, "\sigma(\beta)"'] \arrow[d, "\tau(\alpha)"] & \beta \\
b'' \arrow[ur, dashed, "\overline{\alpha}"] \arrow[r, "\sigma(\alpha)"'] & \alpha &
\end{tikzcd}$$

where the "target" morphisms $\tau(\ \ )$ are well defined like the "source" morphisms $\sigma(\ \ )$, but using ∂_1 instead of ∂_0; but in $\mathcal{F}(B)$ the τ arrows (with label ∂_1) are invertible, hence there are the dotted arrows, so in particular a well defined $b'' \xrightarrow{\overline{\beta}\,\overline{\alpha}} b$ results in $\mathcal{F}(B)$. On the other hand, for any B there is a unique nondegenerate morphism of graphs $B \longrightarrow C =$  to the generic loop (defined by the condition that the fiber be discrete, in other words, that *every* nondegenerate edge of B be labeled by the nondegenerate loop of C). Since $\mathcal{F}$ is a functor

$$\mathcal{S}^{\Delta_1^{op}} \xrightarrow{\mathcal{F}} \mathrm{Cat}$$

we thus get an induced functor $\mathcal{F}(B) \longrightarrow \mathcal{F}(C) \cong \mathbf{N}$ to the additive monoid of natural numbers; this functor is the *length* function on the words in $\mathcal{F}(B)$ (Caution: this length function, like $B \longrightarrow C$ itself, is only functorial with respect to *nondegenerate* morphisms $B_1 \longrightarrow B_2$). For example a graph X over the loop $C =$  in $\mathcal{S}^{\Delta_1^{op}}/C$ satisfies our fibration condition if and only if it is determined by an arbitrary endomap u of the fiber set $X_0 \subset X$, where the rest of edges in X are just the "graph" $\{\langle ux_0, x_0\rangle \mid x_0 \in X_0\}$ of u. Note that usually the free category on B is construed as the skeletal (hence equivalent) subcategory of $\mathcal{F}(B)$ on the objects over 1 in $\overline{\Delta}_1$.

For free categories $\mathcal{F}(B)$ on reflexive graphs, something can occasionally occur (and indeed does in some of our examples such as $B =$ ) which can never occur for the often-studied free categories $W(G)$ on an *irreflexive* G : namely $B \xrightarrow{\sim} \mathcal{F}(B)$ as graphs; of course B would have to be acyclic (i.e. very loop-free) for this to happen.

Since the topos $\mathcal{S}^{\mathcal{F}(B)^{op}}$ is determined as a subcategory of $\mathcal{S}^{\Delta_1^{op}}/B$ via the (epimorphic) functor $\Delta_1/B \xrightarrow{f} \mathcal{F}(B)$, the left-exact idempotent f^*f_* in this case actually preserves even infinite limits, since indeed it has a left adjoint $f^*f_!$ (which is also idempotent)

$$\begin{tikzcd}
\mathcal{S}^{(\Delta_1/B)^{op}} \arrow[r, shift left=2, "f_!"] \arrow[r, shift right=2, "f_*"'] & \mathcal{S}^{\mathcal{F}(B)^{op}} \arrow[l, "f^*"']
\end{tikzcd} \qquad f_! \dashv f^* \dashv f_*$$

Now any free category $\mathcal{F}(B)$ has the property that every morphism in it is both monic and epic. Thus, (as will be explained further) $\mathcal{S}(B) \underset{\text{def}}{=} \mathcal{S}^{\mathcal{F}(B)^{op}}$ serves well as one notion of the "petit topos associated to the object B of the gros topos $\mathcal{B} \underset{\text{def}}{=} \mathcal{S}^{\Delta_1^{op}}$".

Before closing this section, let us illustrate the difference between left actions $\mathcal{S}^{\mathbf{C}}$ and right actions $\mathcal{S}^{\mathbf{C}^{op}}$ in the case $\mathbf{C} = \Delta_1$. In contrast to the infinitely complicated graphs, such left Δ_1, - sets or "cylinders"

$$\begin{tikzcd} A_0 & A \arrow[l, "\mathbf{I}"'] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & {} \end{tikzcd} \qquad \mathbf{I}d_0 = \mathbf{I}d_1 = \mathrm{id}_{A_0}$$

(at least in $\mathcal{S}$!) are all uniquely expressible as disjoint sums of the very special ones in which $A_0 = 1$; these connected cylinders, apart from the cardinality of the fiber A, are determined up to isomorphism by whether or not the two designated points d_0, d_1 are distinct or not.

Note that our cylinders generalize the common notion of cylinders as products as follows: If $1 \rightrightarrows \mathbf{I}$ is any bipointed set (i.e. connected objects of $\mathcal{S}^{\Delta_1}$) and A_0 is any set, then $A \underset{\text{def}}{=} A_0 \times \mathbf{I}$ is an object of $\mathcal{S}^{\Delta_1}$ having the special property that all components are isomorphic. Of course, cylinders *in* a topos other than $\mathcal{S}$ can have highly non-trivial significance; for example when A_0 and A are "contractible" objects of the topos but $d_0(A_0) \cap d_1(A_0) = 0$ in A. In general a $\mathbf{C}^{op}$ action R *in* $\mathcal{S}^{\mathbf{C}}$ gives rise to an adjoint pair of contravariant functors

$$(\mathcal{S}^{\mathbf{C}})^{op} \rightleftarrows \mathcal{S}^{\mathbf{C}^{op}}$$

defined by $\mathrm{Hom}_{\mathbf{C}}(-, {}_1R)$ and $\mathrm{Hom}_{\mathbf{C}^{op}}(-, R_r)$; many of the basic algebra-geometry dualities of mathematics

$$\mathrm{Alg}(\mathbf{C})^{op} \underset{\text{function alg}}{\overset{\text{spectrum}}{\rightleftarrows}} \mathrm{Geom}(\mathbf{C})$$

are just restrictions of exactly such an adjoint pair to subcategories of $(\mathcal{S}^{\mathbf{C}})^{op}$ and $\mathcal{S}^{\mathbf{C}^{op}}$ respectively, with $\mathrm{Geom}(\mathbf{C})$ even being a subtopos defined by a Grothendieck topology on $\mathbf{C}$. A standard example of such R is $R(C', C) = \mathbf{C}(C', C)$, the "hom-functor" of $\mathbf{C}$. However, in the case $\mathbf{C} = \Delta_1$ we could even consider the trivial action on a set R; then any cylinder A yields a special kind of graph

$$R^A \rightleftarrows R^{A_0}$$

which in the case of a connected cylinder ($A_0 = 1$) is just the graph whose *points* are R, whose edges are all functions from A to R, and whose begin and finish vertex relations are given by evaluation

$$f\partial_0 = f(d_0)$$

$$f\partial_1 = f(d_1)$$

at the two points designated by the cylinder structure, for any $f \in R^A$.

As I have mentioned "singularity" several times, it may have occured to the reader that there is an objective way of measuring it. Indeed for any small category $\mathbf{C}$, there is a distinguished object $Eq = Eq_{\mathbf{C}}$ in the topos $\mathcal{S}^{\mathbf{C}^{op}}$ such that for every object X there is a canonical map $X \xrightarrow{\sigma_x} Eq$ in $\mathcal{S}^{\mathbf{C}^{op}}$ which does this. Namely the elements of sort C of Eq are just all the equivalence relations on C, where by an equivalence relation on C is meant the specification for every $D \in \mathbf{C}$ of a set of ordered pairs $D \rightrightarrows C$ of morphisms in $\mathbf{C}$ which is reflexive, symmetric, and transitive for each D and which is closed with respect to composition by arbitrary $D' \longrightarrow D$. If $C' \longrightarrow C$ in $\mathbf{C}$ and $E \in Eq(C)$, then $E \cdot \lambda \in Eq(C')$ is defined by taking for each D

$$\langle t_1, t_2 \rangle \in E \cdot \lambda \Longleftrightarrow \langle \lambda t_1, \lambda t_2 \rangle \in E$$

thus making Eq into an object of $\mathcal{S}^{\mathbf{C}^{op}}$. Of course it is more than just an object, having a natural intersection operation $Eq \times Eq \longrightarrow Eq$ and greatest point $1 \longrightarrow Eq$ making it into a semilattice object, so in particular into an ordered object. On the other hand, although the equality relation $\Delta_C \in Eq(C)$ for each C this is not natural, i.e. does not define a point $1 \xrightarrow{\Delta} Eq$ *unless* it happens that all morphisms in $\mathbf{C}$ are monomorphisms. The singularity measurement $X \xrightarrow{\sigma_x} Eq$ is defined, for each $C \xrightarrow{x} X$, by

$$\sigma(x) = \{\langle t_1, t_2 \rangle \mid xt_1 = xt_2\}$$

the "self-incidence"; then for $C' \xrightarrow{\lambda} C$ we have

$$\sigma(x\lambda) = \sigma(x)\lambda$$

since $(x\lambda)t_1 = (x\lambda)t_2$ iff $x(\lambda t_1) = x(\lambda t_2)$. The maps σ_X, although canonical, are not natural when X is varied, that is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\sigma_X"'] & & Y \arrow[dl, "\sigma_Y"] \\ & Eq & \end{tikzcd}$$

only commutes for "non-singular" f; of course

$$\sigma_X \subseteq \sigma_Y \circ f$$

for all f, so we could say that σ is natural in a suitable "2-categorical" sense. Some f might be "equisingular" in the sense that there exists an endomorphism $|f|$ of Eq so that a square commutes. For example with $\mathbf{C} = \Delta_1$, $Eq_{\mathbf{C}}$ is the loop, and indeed in our generalization $\mathbf{M}(T)$ of Δ_1, Eq itself is very singular.

The reader may have also guessed that Δ_1 is the first of a sequence. The monoid Δ_2 of all order-preserving endomaps of the three-element ordered set $(0, 1, 2)$ has ten elements, three of which (the constants) are points z of the generic figure in that $z\lambda = z$ for all ten λ). The topos $\mathcal{S}^{\Delta_2^{op}}$ may be

considered to consist of triangulated *surfaces,* wherein the triangles may be curled up or singular in other ways and are pasted together along edges or vertices in every conceivable way. A product $X \times Y$ is the 2-skeleton of the 4-dimensional object which one might imagine. The generic triangle has *nineteen* subobjects, so that the "nineteen values of superficial truth" Ω_2 form both a Heyting algebra and compatibly a triangulated surface in which only two of the singular triangles are degenerated all the way to points. If X is any such triangulated surface and $S \subset X$ any subsurface while x is any triangle of X, the statement "$x \in S$" thus has nineteen possible degrees of being false. After having built a model of the surface Ω_2 the reader might like to try $\Omega_3 \ldots$

III. I now will discuss three sequences of examples, assigning to each set T a topos $\mathcal{S}^{\mathbf{M}(T)^{op}}$ which will be gros for $T \geq 2$ and petit toposes $\mathcal{S}^{\mathbf{U}(T)^{op}}$, $\mathcal{S}^{\mathbf{P}(T)^{op}}$. Moreover $\mathcal{S}^{\mathbf{U}(T)^{op}}$ and $\mathcal{S}^{\mathbf{P}(T)^{op}}$ will turn out to be but two instances of a whole system $\mathcal{S}_T(B)$ of petit toposes attached to objects B of $\mathcal{S}^{\mathbf{M}(T)^{op}}$, namely those arising from the special choices $B = B_{\mathbf{U}}$ and $B = B_{\mathbf{P}}$. I will also point out some striking differences between even these simple $\mathcal{S}^{\mathbf{M}(T)^{op}}$ for $T = 1$ versus $T = 2$ versus $T > 2$ versus $T = \infty$. (It is surely relevant that many languages have *four* distinct "numbers": keine, eine, beide, viele.)

The category $\mathbf{U}(T)$ is simply the poset with $T + 1$ elements, in which the added element is greater than all the elements of T and there are no other order relations. This poset indexes a basis of open sets in the topological space which has T isolated points and one "focal" point (whose only neighborhood is the whole space). The presheaf category $\mathcal{S}^{\mathbf{U}(T)^{op}}$ is then the category of *sheaves* on this space, for to specify a sheaf X is merely to specify a set of global sections $X(1)$, a set $X(U_t)$ of sections over each basic open U_t, and restriction maps $X(1) \longrightarrow X(U_t)$ for each $t \in T$. The sheaf condition in this simple case only applies to non-basic open sets

$$U_S = \bigcup_{t \in S} U_t \qquad \text{for } S \subseteq T$$

and then merely forces us to define

$$X(U_S) = \prod_{t \in S} X_t$$

since $U_{t1} \cap U_{t2} = 0$ for $t_1 \neq t_2$; this is the same answer we get if we consider that $U_S \hookrightarrow 1$ is a subobject of the terminal object 1 of $\mathcal{S}^{\mathbf{U}(T)^{op}}$ and define $X(U_S)$ to be the set of (natural, homogeneous) maps $U_S \longrightarrow X$ in $\mathcal{S}^{\mathbf{U}(T)^{op}}$. Everyone seems to agree that toposes constructed in this way from a poset are petit.

On the other hand, the category $\mathbf{P}(T)$, abstracting one simple idea of T parallel processes, is not a poset (unless $T \leq 1$): I define it to have only two objects U and $\mathbf{I}$ but T morphisms from U to $\mathbf{I}$ and no other non-identity

morphisms. Thus, via the Yoneda-Grothendieck-Cayley-Dedekind embedding, the topos $\mathcal{S}^{\mathbf{P}(T)^{op}}$ has a full subcategory

$$\begin{tikzcd} U \arrow[r, bend left=60, bend left=20, bend right=20, bend right=60] & \mathbf{I}. \end{tikzcd}$$

(T arrows)

Figures of type $U \longrightarrow X$ may be called the nodes of X and figures of type $\mathbf{I} \longrightarrow X$ T-gons, so that every T-gon in X has a t-th vertex node for each t in T (some of which may coincide); X is entirely specified by its set of nodes, its set of T-gons, and the t-th vertex relation to each t. For example, if $T = 2$, $\mathbf{P}(2) = \mathbf{P}$ previously alluded to and $\mathcal{S}^{\mathbf{P}(2)^{op}}$ consists of irreflexive graphs, wherein 2-gons are usually called edges and the two kinds of vertices are usually thought of as initial and terminal. Note that the *points* $1 \longrightarrow X$ in $\mathcal{S}^{\mathbf{P}(T)^{op}}$ are exactly the T-gons x whose t-th vertex $xt =$ the same node for all $t \in T$, i.e. generalized "loops".

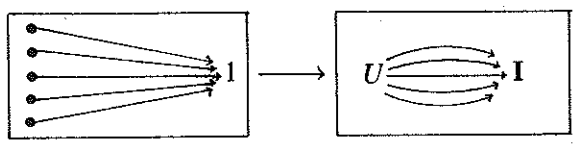
Special interest attaches to the comma category at $\mathbf{I}$

$$\mathcal{S}^{\mathbf{P}(T)^{op}}/\mathbf{I} \approx \mathcal{S}^{\mathbf{P}(T)/\mathbf{I})^{op}}$$

since we can calculate that

$$\mathbf{P}(T)/\mathbf{I} \cong \mathbf{U}(T)$$

which is a poset ! The forgetful $\mathbf{P}(T)/\mathbf{I} \longrightarrow \mathbf{P}(T)$ is just the obvious



Now for any object $\mathbf{I}$ in any topos $\mathcal{E}$, there is a geometric morphism $\mathcal{E}/\mathbf{I} \xrightarrow{\pi} \mathcal{E}$, where the π correctly suggests both "projection" (from a total space to a base) and "product" (internal relative product = object of sections of any $E \longrightarrow \mathbf{I}$). π is just the right adjoint of the obvious functor which assigns

$$I^*F = F_{\mathbf{I}} \underset{\text{def}}{=} \begin{pmatrix} F \times \mathbf{I} \\ \downarrow \text{proj} \\ \mathbf{I} \end{pmatrix}$$

to any F in $\mathcal{E}$. Such a morphism $\mathcal{E}/I \longrightarrow \mathcal{E}$ is often considered to be a "local homeomorphism" since its inverse image functor $(\ \ )_{\mathbf{I}}$ preserves all the internal higher-order logic of $\mathcal{E}$:

$$(X \times Y)_{\mathbf{I}} = X_I \times Y_I = \text{product in } \mathcal{E}/\mathbf{I}$$

$$(Y^A)_{\mathbf{I}} = Y_{\mathbf{I}}^{A_{\mathbf{I}}} = \text{function space in the sense of } \mathcal{E}/\mathbf{I}$$

$$\Omega_{\mathbf{I}} = \text{the truth-value object of } \mathcal{E}/\mathbf{I} \text{ etc.}$$

If moreover the object $\mathbf{I}$ is a *covering* of the topos $\mathcal{E}$ in the sense that the unique map $\mathbf{I} \longrightarrow 1$ is an epimorphism of $\mathcal{E}$, then $\mathcal{E}/I \longrightarrow \mathcal{E}$ has the property

of being "conservative", i.e. f is a monomorphism in $\mathcal{E}$ iff $f_{\mathbf{I}}$ is a monomorphism in $\mathcal{E}/\mathbf{I}$, f is a epimorphism in $\mathcal{E}$ iff $f_{\mathbf{I}}$ is an epimorphism in $\mathcal{E}/\mathbf{I}$, etc. Then one says that a topos $\mathcal{E}$ "locally" has a certain property if there exists a covering $\mathbf{I} \twoheadrightarrow 1$ of $\mathcal{E}$ such that $\mathcal{E}/\mathbf{I}$ actually has the property. It seems eminently reasonable that a topos which is locally petit should also be considered petit; at least, this seems to be implicit in Grothendieck's concept of *étendue* (which he now calls *topological multiplicity*): this refers to any topos $\mathcal{E}$ which is *locally* a subtopos of a presheaf topos $\mathcal{S}^{\mathbf{U}^{op}}$ where $\mathbf{U}$ is some *poset*. If $\mathbf{C}$ is any category in which every morphism is a monomorphism (for example any monoid satisfying the cancellation law $ax = ay \Rightarrow x = y$), then $\mathcal{S}^{\mathbf{C}^{op}}$ is an étendue, since it is locally presheaves on the poset $\mathbf{C}/\Sigma C$.

It is obvious that every morphism in $\mathbf{P}(T)$ is a monomorphism, since there are no non-trivial composites $ax = ay$ to check, and anyway we have already constructed the surjective local homeomorphism $\mathbf{U}(T) \longrightarrow \mathbf{P}(T)$ from a poset. Thus we are justified in considering that $\mathcal{S}^{\mathbf{P}(T)^{op}}$ is petit. On the other hand, it is not gros, for although the components functor $\mathcal{S}^{\mathbf{P}(T)^{op}} \longrightarrow \mathcal{S}$ takes Ω_T to 1 for $T \geq 2$, for the same T's it *fails* to preserve finite products.

The generic node U in $\mathcal{S}^{\mathbf{P}(T)^{op}}$ is not a covering, since in fact it is the unique non-trivial one among the three subobjects of 1 (thus Ω_T has three generalized "loops" $1 \longrightarrow \Omega_T$); $\mathcal{S}^{\mathbf{P}(T)^{op}}/U \cong \mathcal{S}$ since if $X \longrightarrow U$ then $(\mathbf{I}, X) = 0$ for $(\mathbf{I}, U) = 0$; thus X consists only of nodes $U \longrightarrow X$, each of which is a section of $X \longrightarrow U$; the map $X \longrightarrow U$ is unique if it exists since $U \hookrightarrow 1$; therefore $X = S \times U$ where $S = (U, X)$ is considered discrete. In particular the restriction of the object $\mathbf{I}$ to this open subtopos $\mathcal{S} \cong \mathcal{S}^{\mathbf{P}(T)^{op}}/U \hookrightarrow \mathcal{S}^{\mathbf{P}(T)^{op}}$ is the set T; in other words $U \times \mathbf{I} = T \times U$ with T considered discrete.

For the object $\mathbf{I}$ of $\mathcal{S}^{\mathbf{P}(T)^{op}}$, we have $\mathbf{I}^2 = \mathbf{I} + (T^2 - T) \times U$, where $T^2 - T$ is considered as a discrete object; for $(U, \mathbf{I}^2) = (U, \mathbf{I})^2 = T^2$ is the number of nodes of $\mathbf{I}^2$, whereas $(\mathbf{I}, \mathbf{I}^2) = (\mathbf{I}, \mathbf{I})^2 = 1^2 = 1$ is the number of T-gons of $\mathbf{I}^2$, and this unique element must be the *diagonal* $\mathbf{I} \longrightarrow \mathbf{I}^2$ which is nonsingular and hence have T distinct nodes, leaving $T^2 - T$ bare nodes in $\mathbf{I}^2$. Combining the above quadratic equation with the two equations $U^2 = U$, $U \times \mathbf{I} = T \times U$ we get a presentation of a small but significant part of the Burnside half-ring of the topos $\mathcal{S}^{\mathbf{P}(T)^{op}}$; significant because it generates the whole topos with the help of colimits, small because using only the half-ring operations, $\mathbf{I}$ and U don't generate any very complicated objects. This sub-half-ring has a very simple description after tensoring with the rationals $\mathbf{Q}$ or even with $\mathbf{Z}[1/T^2 - T]$, i.e. after adjoining negatives and sufficiently many denominators (here we assume $T > 1$ is finite, and treat $T^2 - T$ as the natural number which is its cardinality): For let x denote the element of this rationalized Burnside ring which corresponds to the generic figure $\mathbf{I}$, and similarly let u correspond to the generic node U; then the quadratic equation implies that the generic node can be expressed in terms of the generic figure

$$u = (x^2 - x)/(T^2 - T)$$

so that the subring in question is in fact generated (over the scalars $\mathbf{Z}[1/T^2 - T]$ by the single element x. However x satisfies the relations $xu = Tu$ where T is a whole number and also $u^2 = u$. Writing out the first of these we get

$$x^3 = (1+T)x^2 - Tx$$

and hence $x^4 = (1+T+T^2)x^2 - (1+T)Tx$. Then by multiplying out u^2 in terms of x we find that the second relation $u^2 = u$ actually follows from the first. On the other hand the cubic equation actually factors as

$$\boxed{x(x-1)(x-T) = 0}$$

Thus we have proved that

THEOREM. *The subring of the rationalized Burnside ring of* $\mathcal{S}^{\mathbf{P}(T)^{op}}$ *which is generated by the Yoneda embedding is actually the ring of functions on the three-point spectrum* $(0, 1, T) \subset \mathbf{Z}$.

This information helps to determine the points of the topos, $\mathcal{S} \longrightarrow \mathcal{S}^{\mathbf{P}(T)^{op}}$ whose inverse image functors are required to be left exact and cocontinuous, and hence correspond to *left* actions of the category $\mathbf{P}(T)$ which are "flat": namely the latter can only have the three possible cardinalities (for their set of "quantities" of type $\mathbf{I}$) $0, 1, T$. It would be interesting to find such an algebraic presentation of a portion including Ω of the (rationalized) Burnside ring of $\mathcal{S}^{\mathbf{P}(T)^{op}}$.

By $\mathbf{M}(T)$ I will mean the monoid which has $T+1$ elements, the added element being the identity element, with the multiplication law

$$\boxed{ts = t} \qquad t \in T, s \in \mathbf{M}(T).$$

Not only are all elements of $M(T)$ idempotent, but any pair of elements satisfies the following identity

$$\boxed{\alpha\beta\alpha = \alpha\beta}.$$

Such "graphic" monoids are relevant to the study of lists without repetitions and to "check-in" actions: if $\mathbf{M}$ is any graphic monoid and X is any right action of $\mathbf{M}$ on a set of states, then when α checks in, it may change the state, but if he tries later to check in a second time it is surely irrelevant. Not all graphic monoids $\mathbf{M}$ are of the simple form $\mathbf{M}(T)$, but there is a structure theorem for them saying that if we force commutativity $\mathbf{M} \xrightarrow{\sigma} \mathbf{S}$ we get a sup-semilattice with a support function

$$(\sigma(1) = 0, \sigma(\alpha\beta) = \sigma(\alpha) \cup \sigma(\beta))$$

with the property that if we consider $T_a = \{\alpha \mid \sigma(\alpha) = a\}$, the set of all elements with support equal to given $a \in \mathbf{S}$, then we get a homomorphic inclusion $\mathbf{M}(T_a) \longrightarrow \mathbf{M}$ for $a \neq 0$. Thus it is tempting to recast the structure theory in topos-theoretic terms by replacing the base topos $\mathcal{S}$ by another one

constructed using the semilattice **S**, but I have not yet had time to work this out.

The idea to consider $\mathbf{M}(T)$ was reinforced by a remark of Professor Jorge Gracia in a discussion of the philosophical part/whole relation: the mere *idea* that T points cohere into a single whole could be considered totally abstractly as just another element adjoined to the abstract set T. Then much more complex wholes could be analyzed as being made of overlapping parts (possibly singular) of this simple kind. For example the idea of connecting two points ($T = 2$) may be expressed by the three element figure $\boxed{\bullet \longrightarrow \bullet}$. In a complex graph X, a given pair of points $2 \longrightarrow X$ (where $2 = 1 + 1$ is in itself discrete) may or may not be connectable

$$\begin{tikzcd} \boxed{\bullet \quad \bullet} \arrow[r, hook] \arrow[dr] & \boxed{\bullet \longrightarrow \bullet} \arrow[d, dashed] \\ & X \end{tikzcd}$$

and if they are connectable it may be possible in many ways. So it is also for arbitrary T. Note that (because of the draconian multiplication table), $1 \in \overline{\mathbf{M}(T)}$, and indeed $\overline{\mathbf{M}(T)}$, the category obtained from the monoid by splitting idempotents, is equivalent to a category with only two objects, the terminal object and the one represented by the unique object of the monoid:

$$\begin{tikzcd} 1 \arrow[r, bend left, "T"] \arrow[r, bend left=20] \arrow[r, bend left=40] & \mathbf{I} \arrow[l, bend left, "\mathbf{I}"] \arrow[r, bend left, "T+1"] \arrow[r, bend left=20] \arrow[r, bend left=40] & \end{tikzcd}$$

All endomaps of **I** except the identity $1_{\mathbf{I}}$ are constant; in other words, (for example $\mathbf{M}(2) = \Delta_1$ as discussed in section II): The discrete object

$$T = \sum_T 1$$

is a subobject $T \subset \mathbf{I}$ in $\mathcal{S}^{\mathbf{M}(T)^{op}}$ which contains all points of **I**, but which is very different from **I**, for it lacks the breath of cohesiveness which **I** has and which makes all the difference although this restrictive method temporarily neglects further analysis of this unity. For any object X of $\mathcal{S}^{\mathbf{M}(T)^{op}}$ and any T-indexed family of points $1 \xrightarrow{x_t} X$ of X there is (by the universal mapping property of Σ) a single morphism $T \xrightarrow{x} X$ and hence a set (possibly empty) of (possibly singular) **I**-figures $\overline{x}$

$$\begin{tikzcd} T \arrow[dr, "x"'] & \subset & \mathbf{I} \arrow[d, dashed, "\overline{x}"] \\ & & X \end{tikzcd}$$

in X whose vertices agree $\overline{x}\cdot t = x_t$ for all t with the given points. The resulting myriads of incidence relations are preserved by any morphism $X \longrightarrow Y$.

By looking at the subobjects of $\mathbf{I}$ in $\mathcal{S}^{\mathbf{M}(T)^{op}}$ we see that there are precisely

$$2^T + 1$$

truth-values in the object Ω_T. The Heyting algebra thus resulting is also "simplest possible" in that the Boolean property

$$A = \neg\neg A$$

is violated for precisely one A, namely $A = T$, for $\neg\neg T = \mathbf{I} > T$, while it holds for $A = \mathbf{I}$ or for any $A \subseteq T$ with $A \neq T$.

For any T the components functor (left adjoint to the discrete inclusion) $\mathcal{S}^{\mathbf{M}(T)^{op}} \longrightarrow \mathcal{S}$ may be computed as the *reflexive* coequalizer

$$(T^2 \times \mathbf{I}, X) \rightrightarrows (1, X) \longrightarrow \pi_0(X)$$

and hence preserves finite cartesian products. For $T > 1$, there is a pair $1 \rightrightarrows \mathbf{I}$ of morphisms with empty intersection, thus (as was pointed out by Grothendieck), if we consider the map $\mathbf{I} \longrightarrow \Omega$ corresponding to one of them (which maps it to true), the others are mapped to false; this has the effect that all non-zero elements of Ω can be connected (by the action of $\mathbf{M}(T)$ on Ω, which is by inverse image) to zero and hence that $\pi_0(\Omega) = 1$. For any T, the points functor $\mathcal{S}^{\mathbf{M}(T)^{op}} \longrightarrow \mathcal{S}$ has the right adjoint sending any set S to S^T, providing a notion of chaotic object opposite to discrete. Thus for $T > 1$, $\mathcal{S}^{\mathbf{M}(T)^{op}}$ satisfies three of the distinctive properties of a "gros" topos.

For $T = 1$, $\mathbf{M}(T) = \{0, 1\}$ the multiplicative monoid with one idempotent $0 \neq 1$. We have that Ω has $2^1 + 1 = 3$ elements, but $\mathbf{M}(1)$ lacks sufficient action to connect T, $\mathbf{I}$ with 0, hence $\pi_0(\Omega) = 2$. Thus it seems that $\mathcal{S}^{\mathbf{M}(1)^{op}}$ is not gros. Probably it is not petit either; at least the $\mathbf{M}(1)/\mathbf{I}$ construction fails to produce a poset.

What possible use could the categories $\mathcal{S}^{\mathbf{M}(T)^{op}}$ for $T > 2$ have? As a theory of triangulated surfaces, $\mathcal{S}^{\mathbf{M}(3)^{op}}$ seems much poorer than the category $\mathcal{S}^{\Delta_2^{op}}$ discussed briefly in section II, since the six non-constant, non-identity operators in Δ_2 permit explicit calculation of the incidences of the *boundaries* of the triangles, not just of the vertices as here.

If T is a countably infinite set with a distinguished point ∞, then we can define a functor

$$\mathrm{top} \longrightarrow \mathcal{S}^{\mathbf{M}(T)^{op}}$$

from the category of topological spaces and continuous maps, by using the distinguished point as follows: we topologize T by making every $t \neq \infty$ an open point and by making every *cofinite* subset containing ∞ into a neighborhood of ∞ resulting in a space denoted T_∞. Then for any space X, the set $\mathrm{top}(T_\infty, X)$ of continuous maps $T_\infty \longrightarrow X$ is the set of *convergent sequences* in X, which has the obvious action of $\mathbf{M}(T)$ defined by $x \cdot t =$ the *constant*

sequence which is constantly equal to the t-th term of the sequence x; in particular

$$\boxed{x_\infty = \lim_{t \neq \infty} x_t}.$$

Clearly

$$\begin{tikzcd} \mathrm{top} \arrow[rr, "\mathrm{top}(T_\infty, -)"] \arrow[dr, "\mathrm{points}"'] & & \mathcal{S}^{\mathbf{M}(T)^{op}} \arrow[dl, "\mathrm{points}"] \\ & \mathcal{S} & \end{tikzcd}$$

commutes, so that in particular our functor is faithful and for an object of $\mathcal{S}^{\mathbf{M}(T)^{op}}$ coming from top, the above limit condition is necessary and sufficient for a morphism x from the discrete T to extend to a (unique) cohesive figure

$$\begin{tikzcd} T \arrow[dr] & \subset & \mathbf{I} \arrow[dl, dashed] \\ & X & \end{tikzcd}$$

That is, cohesive means convergent for such objects. If we restrict our functor to the subcategory top_ω of sequentially determined spaces (which includes all metrizable spaces) our functor (which in any case preserves products and equalizers) becomes *full*; this is due to the fact that preservation of convergent sequences is sufficient to determine continuity in top_ω, whereas the naturality/homogeneity of the maps in $\mathcal{S}^{\mathbf{M}(T)^{op}}$ means precisely that convergent sequences are mapped to convergent sequence, preserving the evaluation at *each* t. The inclusion $\mathrm{top}_\omega \hookrightarrow \mathcal{S}^{\mathbf{M}(T)^{op}}$ has a left adjoint, and preserves the function space construction (which exists in top_ω too). Thus we are justified in considering top_ω as a full subcategory of $\mathcal{S}^{\mathbf{M}(T)^{op}}$ (of course for a different choice of $\infty \in T$ we would get a different subcategory). Note that $\mathbf{I}$ is *not* in the subcategory; the space T_∞ determines a much bigger object

$$T \hookrightarrow \mathbf{I} \hookrightarrow T_\infty$$

(all with the same points) but of course for each X in $\mathcal{S}^{\mathbf{M}(T)^{op}}$ which comes from a space, every $\mathbf{I} \longrightarrow X$ uniquely extends to $T_\infty \longrightarrow X$. The Sierpinski space 2 (two points, three open sets) determines a Heyting algebra object $\mathbf{O} = \mathrm{top}(T_\infty, 2)$ of $\mathcal{S}^{\mathbf{M}(T)^{op}}$ with a morphism $\mathbf{O} \longrightarrow \Omega$ given by restricting from T_∞ to $\mathbf{I}$; this suggests considering $\mathbf{O}^X$ as the "object of opens" for any object X of $\mathcal{S}^{\mathbf{M}(T)^{op}}$. However, we are not completely free to use $\mathcal{S}^{\mathbf{M}(T)^{op}}$ (with its super-simple definition) as a replacement (improvement in that it is a topos) for top_ω, because topological sums are not preserved by the embedding: there is the comparison map

$$\mathrm{top}(T_\infty, X_1) + \mathrm{top}(T_\infty, X_2) \longrightarrow \mathrm{top}(T_\infty, X_1 + X_2)$$

(which is bijective on points) in the topos, but the "convergent sequences" in the presheaf-sum are either completely in one summand or completely in the other, whereas in the top_ω-sum on the right a convergent sequence may

bounce back and forth any finite number of times before finally settling into either X_1 or X_2 where its limit is.

The fact that cohesiveness in $\mathcal{S}^{\mathbf{M}(T)^{op}}$ (for T countably infinite) is not necessarily of the "limit" kind is illustrated by a completely different cartesian-closed full reflective subcategory bor_ω obtained as follows. Bornological sets are usually defined to consist of a family of "bounded" subsets of a given set of points, subject to the axioms that all singletons are bounded, any subset of a bounded set is bounded, and the union of any finite collection of bounded sets is bounded. A morphism from one bornological set to another is any mapping of points for which the image of any bounded set in the domain is bounded in the codomain. Thus we get a (cartesian closed) category bor. A bornological set is discrete iff only the finite subsets are bounded, the opposite of "chaotic" in which all subsets are bounded; the structure of a bornological set is determined by knowing the chaotic figures in it. This seems to be a very loose structure, but in combination with algebraic structure it is quite important in functional analysis, where its covariant nature is much easier to deal with than the contrary "open set" determination: for example, the category ab(bor) of bornological abelian groups contains the usual categories of Frechet nuclear spaces, Banach spaces, etc. with continuous linear maps as *full* subcategories. Let T_{bor} be the codiscrete bornological set with T points. Then

$$\mathrm{bor} \xrightarrow{\mathrm{bor}(T_{\mathrm{bor}},-)} \mathcal{S}^{\mathbf{M}(T)^{op}}$$

again preserves the natural point functors on both categories and has a left adjoint. If bor_ω is defined as the subcategory of those bornological sets for which any subset is bounded provided every countable subset of it is bounded, then the restriction of our functor to bor_ω is again full and has many good properties, but again fails to preserve sums. We get a disparate big enlargement of $\mathbf{I}$ with the same points

$$T \subset \mathbf{I} \begin{array}{l} \subset T_{\mathrm{bor}} \\ \subset T_\infty \end{array}$$

in $\mathcal{S}^{\mathbf{M}(T)^{op}}$ such that $(T_{\mathrm{bor}}, X) \xrightarrow{\sim} (\mathbf{I}, X)$ is a bijection for bornological objects X. This is the largest extension $\mathbf{I} \subset E$ of $\mathbf{I}$ which has the same points $T = (1, \mathbf{I}) \xrightarrow{\sim} (1, E)$ *and* which is separated by 1 in the sense that for any $\mathbf{I} \underset{g}{\overset{f}{\rightrightarrows}} E$, $f = g$ provided the induced mappings $(1, \mathbf{I}) \rightrightarrows (1, E)$ are equal, i.e. $\forall t \in T[ft = gt] \Rightarrow f = g$, i.e. $E \in \mathcal{Y}_{\mathbf{M}(T)}$.

It is even possible to get embeddings of topological categories into $\mathcal{S}^{\mathbf{M}(T)^{op}}$ which do preserve sums, provided we take T to be the power of the continuum and imagine it topologized as an interval or as a circle, since either of the latter is a *connected* space. For then

$$\mathrm{top}(T_{\mathrm{cts}}, \sum_i X_i) \xleftarrow{\sim} \sum_i \mathrm{top}(T_{\mathrm{cts}}, X_i)$$

in $\mathcal{S}^{\mathbf{M}(T)^{op}}$. Thus a cohesive figure is now interpreted to mean a continuous path, or a continuous loop, etc., depending on the meaning fixed for T_{cts}; the embedding will be *full* when restricted to locally arcwise-connected spaces. Such is the unexpected power of the action of constant maps.

Returning to the finite, let us consider briefly some of the remarkable properties of the apparently simple topos $\mathcal{S}^{\mathbf{M}(1)^{op}} = \mathcal{S}^{\{0,1\}^{op}}$ consisting of sets operated on by a single idempotent. Even though it is not gros, it is connected to any of our $\mathcal{S}^{\mathbf{M}(T)^{op}}$ through the homomorphism $\mathbf{M}(T) \longrightarrow \{0,1\}$ which collapses all T to 0, as well as by its T different sections $\{0,1\} \hookrightarrow \mathbf{M}(T)$. Restriction along the latter ones, and the left and right adjoints to restriction along the former one thus yield $T+2$ functors $\mathcal{S}^{\mathbf{M}(T)^{op}} \longrightarrow \mathcal{S}^{\{0,1\}^{op}}$ which may provide useful invariants for classifying the objects of $\mathcal{S}^{\mathbf{M}(T)^{op}}$, especially since most of these functors preserve both sums and products; especially for graphs ($T = 2$) one is always thirsty for systematic information. These invariants may be construed as quantitative in the Galileo-Cantor-Burnside-Grothendieck spirit as follows: recall that every X in $\mathcal{S}^{\{0,1\}^{op}}$ is uniquely expressible as a sum of connected components and that a connected object is just a pointed set. If we let B_n denote a standard pointed set with n elements, and $\hat{X}(n)$ denote the abstract set of components of type B_n, then we can write

$$X = \sum_n \hat{X}(n) \times B_n$$

(Incidentally $\sum_n \hat{X}(n)$ is the set of *points* of X, since B_n has a unique point). Now, since

$$B_n \times B_m \cong B_{n \times m}$$

we can compute cartesian products in terms of the above expansion as follows, using the fact that any topos is a distributive category:

$$\begin{aligned} X \times Y &= \Big(\sum_n \hat{X}(n) \times B_n\Big) \times \Big(\sum_m \hat{Y}(m) \times B_m\Big) \\ &= \sum_n \sum_m \hat{X}(n) \times \hat{Y}(m) \times (B_n \times B_m) \\ &= \sum_q \Big[\sum_{n \times m = q} \hat{X}(n) \times \hat{Y}(m)\Big] \times B_q \end{aligned}$$

This is just the rule for multiplying *formal Dirichlet series,* which are usually written with B_n replaced by $1/(n)^s$ where $1/(\ )^s$ is a formal character of the situation but for which one sometimes succeeds in evaluating $X(s)$ for some complex numbers s provided the coefficients $\hat{X}(n)$ comply. Now when the sets $\hat{X}(n)$ are finite, and non-zero only for finite n, we can via cardinality interpret the coefficients as whole numbers. This gives an explicit determination of the *Burnside ring* $\mathcal{R}(1)$ of the topos $\mathcal{S}^{\mathbf{M}(1)^{op}}$, which is defined to consist of all isomorphism types of objects satisfying the stated finiteness conditions, added and multiplied by categorical coproduct and product, with differences formally adjoined. (One of the main reasons for the finiteness conditions is to

ensure that these "virtual" differences do not by their introduction produce collapse.) The Burnside rings were originally found useful in the case $\mathcal{S}^G$ of the topos of permutation representations of a group G. The $T+1$ ring homomorphisms and one linear operator

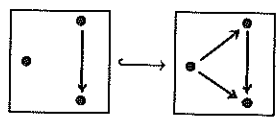
$$\mathcal{R}(T) \rightrightarrows \mathcal{R}(1)$$

thus assign to each $X \in \mathcal{S}^{\mathbf{M}(T)^{op}}$ (for example each graph when $T = 2$) a system of "ζ-functions" in the ring of formal Dirichlet series with $\mathbf{Z}$ coefficients. The classical Riemann ζ-function corresponds to the unique object of $\mathcal{S}^{\{0,1\}^{op}}$ which has exactly one component B_n of size n for every finite n.

To study a little more closely the differences between the toposes $\mathcal{S}^{\mathbf{M}(T)^{op}}$ for various T, consider the problem of computing the function space $\mathbf{I}^{\mathbf{I}}$, where $\mathbf{I}$ is the regular right representation. In general there is $1 \longrightarrow \mathbf{I}^{\mathbf{I}}$, the name of the identity map $\mathbf{I} \longrightarrow \mathbf{I}$, and also a map $\mathbf{I} \longrightarrow \mathbf{I}^{\mathbf{I}}$, the inclusion of constants coming as a special case of the $Y \longrightarrow Y^{\mathbf{I}}$ induced by the unique $\mathbf{I} \longrightarrow 1$. Putting these two together we have the map

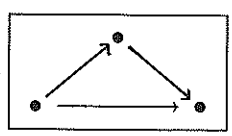
$$1 + \mathbf{I} \longrightarrow \mathbf{I}^{\mathbf{I}},$$

as for any object in any topos. This map is sometimes an isomorphism, but certainly not in $\mathcal{S}^{\mathbf{M}(2)^{op}} = \mathcal{S}^{\mathbf{A}_1^{op}}$, where from our computation in section II, we see that $1 + \mathbf{I} \hookrightarrow \mathbf{I}^{\mathbf{I}}$ is the map whose internal picture is



(The reader may wish as an exercise to compute the characteristic map $\mathbf{I}^{\mathbf{I}} \longrightarrow \Omega$ to the five-edge truth-value object for this inclusion.) On the other hand, for $T > 2$, the map $1 + \mathbf{I} \longrightarrow \mathbf{I}^{\mathbf{I}}$ *is* an isomorphism in $\mathcal{S}^{\mathbf{M}(T)^{op}}$; that is, all maps $\mathbf{I} \times \mathbf{I} \xrightarrow{f} \mathbf{I}$, (in other words all functions of two variables on the monoid to itself satisfying $f(ut, vt) = f(u, v)t$) are of a very simple form, as Steve Schanuel showed me.

Now $1 + x$ is a very simple polynomial, and we have said that for $T > 2$ the exponential $\mathbf{I}^{\mathbf{I}}$ is in fact that polynomial applied to $\mathbf{I}$. On the other hand in the category of graphs with $T = 2$, we have

$\mathbf{I}^{\mathbf{I}} =$ 

which is manifestly neither a sum nor a product and hence not any polynomial function of $\mathbf{I}$.

THEOREM. *For any $T \neq 2$, there is a polynomial Φ_T with $\mathbf{N}$ coefficients such that for the representable object $\mathbf{I} \in \mathcal{S}^{\mathbf{M}(T)^{op}}$ one has*

$$\mathbf{I}^{\mathbf{I}} = \Phi_T(\mathbf{I})$$

In fact $\Phi_T(x) = 1 + x$ *for all* $T > 2$, *whereas* $\Phi_0 = 1$ *and*

$$\Phi_1(x) = 2x^2.$$

To understand this further surprise from $\mathcal{S}^{\{0,1\}^{op}}$, note that for any connected B_n, $B_n^{\mathbf{I}}$ has (like any object) as many components as points, but its points correspond to morphisms $\mathbf{I} \longrightarrow B_n$, which are n in number. The number of general elements $\mathbf{I} \longrightarrow B_n^{\mathbf{I}}$ in $B_n^{\mathbf{I}}$ is the same as the number of morphisms $\mathbf{I}^2 \longrightarrow B_n$, so we need to consider $\mathbf{I}^2$; it has just one point, but $(\mathbf{I}, \mathbf{I}^2) = (\mathbf{I}, \mathbf{I})^2 = 2^2 = 4$ elements, so $\mathbf{I}^2 = B_4$. A map $B_4 \longrightarrow ?$ must map the point to a point but can map the other three elements arbitrarily. Hence $B_n^{\mathbf{I}}$ has n^3 elements, $n^3 - n$ of which are non-degenerate. If we could determine the apportionment of these $n^3 - n$ elements among the n components, we could essentially compute $Y^{\mathbf{I}}$ for any Y, since $\mathbf{I}$ is so strongly connected that

$$Y^{\mathbf{I}} = \Big(\sum_n \hat{Y}(n) \times B_n\Big)^{\mathbf{I}} = \sum_n \hat{Y}(n) \times B_n^{\mathbf{I}}$$

This apportionment for n a power of 2 follows from the case $B_2 = \mathbf{I}$. To determine how the $2^3 - 2 = 6$ nondegenerate elements of $\mathbf{I}^{\mathbf{I}}$ are apportioned among the two components, recall that the right action of T on these elements in their $\mathbf{I}^2 \xrightarrow{f} \mathbf{I}$ guise is by multiplying on the left only in the first variable

$$(ft)(s, x) = f(ts, x).$$

Hence for $T = \{0\}$ we get

$$\begin{aligned}(f \cdot 0)(1,1) &= f(0,1)\\ (f \cdot 0)(1,0) &= f(0,0)\\ (f \cdot 0)(0,1) &= f(0,1)\\ (f \cdot 0)(0,0) &= f(0,0)\end{aligned}$$

Thus, for any $\mathbf{I}^2 \xrightarrow{f} \mathbf{I}$, $f \cdot 0$ factors across the second projection $\mathbf{I}^2 \longrightarrow \mathbf{I}$, so is a *point* of $\mathbf{I}^{\mathbf{I}}$. The second projection itself corresponds to the point 1 of $\mathbf{I}^{\mathbf{I}}$, whereas the constantly 0 map $\mathbf{I}^2 \longrightarrow \mathbf{I}$ corresponds to the point 0 of $\mathbf{I}^{\mathbf{I}}$. Further consideration of cases shows that among the six f which do not factor across the second projection, three of them have $f \cdot 0 = 1$ and three of them have $f \cdot 0 = 0$. Hence the two components are of equal size $B_4 = \mathbf{I}^2$. In other words $\mathbf{I}^{\mathbf{I}} = 2\mathbf{I}^2$, as was to be shown.

[The exceptional behavior of $\mathcal{S}^{\mathbf{M}(T)^{op}}$ in the special case $T = 1$ may perhaps be explained by noting that since the monoid $\{0, 1\}$ is commutative, we could equally well consider it to be one of the sequence $\mathcal{S}^{\mathbf{M}(T)}$ of toposes of *left* actions, i.e. of "cylinders" which have T preferred sections instead of just two. All of these toposes (for $T \neq 0$) have *three* truth values Ω with two components, and all of them in fact have the components functor representable. The representing object for π_0 is T, the (*connected* this time) subobject of the generic object, which itself has no non-empty subobjects: $\pi_0 A = \mathcal{S}^{\mathbf{M}(T)}(T, A)$

for all objects A, so in particular T has only the identity endomorphism (as opposed to T^T in the case of right actions). The connected objects of $\mathcal{S}^{\mathbf{M}(T)}$ are parameterized by the category $T/\mathcal{S}$ of T-pointed sets, and are hence determined by the cardinality (of the set of figures, not "points") together with an equivalence relation on T. The Burnside ring is thus the tensor product of the formal Dirichlet series with the formal monoid ring of the monoid of all equivalence relations on T with intersection of equivalence relations as multiplication. The object A^A for A the generic object can be computed by a formula generalizing that for the case $T = 1$.]

The Theorem has the consequence that (for $T \neq 0$), the object $\mathbf{I}$ is *contractible* in $\mathcal{S}^{M(T)^{op}}$ iff $T = 2$.

Now it remains to discuss the relation between the gros toposes $\mathcal{S}^{\mathbf{M}(T)^{op}}$ (for $T \geq 2$) and the two sequences $\mathcal{S}^{\mathbf{U}(T)^{op}} \longrightarrow \mathcal{S}^{\mathbf{P}(T)^{op}}$ of petit toposes (actually étendue in these cases). I will do this by attaching to each object of the gros topos an associated petit topos by a reduction of the associated "comma category" or discrete fibration. That is, for suitable $\mathbf{C}$, (which if "big" enough will give a gros $\mathcal{S}^{\mathbf{C}^{op}}$) we will show that

$$\begin{tikzcd}
\mathcal{S}^{\mathbf{C}^{op}} \arrow[r, "\mathbf{C}/-"] \arrow[d, dashed] & \mathrm{Cat} \arrow[d, shift right, "\widehat{(\,)}"'] \arrow[dr, "\mathcal{S}^{(\,)^{op}}"] & \\
\mathrm{Cat}_{\mathrm{mono}} \arrow[r, hook] \arrow[rr, bend right] & \mathrm{Cat}_{\mathrm{red}} \arrow[u, hook, shift right] & \mathrm{Top}
\end{tikzcd}$$

in other words that the reduction $\widehat{\mathbf{C}/B}$ of $\mathbf{C}/B$ for any $B \in \mathcal{S}^{\mathbf{C}^{op}}$ actually is a category whose topos of right actions is guaranteed to be petit, since $\widehat{\mathbf{C}/B}$ itself consists entirely of monomorphisms. For every $B_1 \xrightarrow{f} B_2$ in $\mathcal{S}^{\mathbf{C}^{op}}$, the resulting map of petit toposes is automatically "essential", that is we will always have the left Kan quantifier $f_!$ as well as the inverse image f^* and the right Kan quantifier f_*.

The meaning of "reduced" categories which will turn out to suffice for our limited purpose is

$$e \circlearrowleft C \;\&\; e^2 = e \Rightarrow e = 1$$

in other words there are no idempotents (except identities). Since most of our examples $\mathbf{C}$ consist "mainly" of idempotents and become $\mathbf{C}/B$ with the same feature, reducing $\mathbf{C}/B$ to $\widehat{\mathbf{C}/B}$ by collapsing the idempotents records principally the "change" in the "becoming".

Like any class of categories closed under arbitrary product and arbitrary subcategories, the inclusion $\mathrm{Cat}_{\mathrm{red}} \hookrightarrow \mathrm{Cat}$ has a left adjoint with surjective adjunction functors

$$\mathbf{C} \longrightarrow \widehat{\mathbf{C}}$$

which in this case are bijective on objects so determined by a categorical congruence relation, i.e. an equivalence relation on each set $\mathbf{C}(C', C)$, stable under composition. In our case the equivalence relation starts off being generated by all pairs $\langle e_1, e_2 \rangle \in \mathbf{C}(C, C)$ for which both e_1, e_2 are idempotents.

This indicates the meaning of my reduction process ($\hat{}$). It may be noted that $\widehat{\mathbf{C}}$ may also be described as the result of formally inverting all split epimorphisms (i.e. all p for which there exists a section s, $ps = id$) in case all idempotents split in $\mathbf{C}$. (In any case, this particular "fraction" construction, unlike most gives a *surjective* functor $\mathbf{C} \longrightarrow \widehat{\mathbf{C}}$). The reduction preserves finite products of categories.

Of course very strong conditions on $\mathbf{C}$ are required in order to conclude that all maps become monomorphisms in $\widehat{\mathbf{C}}$. Since I don't know a very explicit description in general of the congruence relation involved in $\mathbf{C} \longrightarrow \widehat{\mathbf{C}}$, I will need very strong conditions indeed. It would be very convenient (also in other applications such as simplicial sets) if one knew what conditions (if any) must be added to

> "Every map in $\mathbf{C}$ can be factored as a split epimorphism followed by a monomorphism"

in order to conclude that every map in $\widehat{\mathbf{C}}$ is a monomorphism; this comes down to whether $\mathbf{C} \longrightarrow \widehat{\mathbf{C}}$ *preserves* monomorphisms, since certainly any map in $\widehat{\mathbf{C}}$ is isomorphic to one of the form $[i]$, the congruence class of a map i which is a monomorphism in $\mathbf{C}$.

The condition in quotation marks above was carefully chosen to be stable under the passage from $\mathbf{C}$ to $\mathbf{C}/B$ where B is any object of $S^{\mathbf{C}^{op}}$, or equivalently to be stable under the passage from $\mathbf{C}$ to a category $\mathbf{E} \longrightarrow \mathbf{C}$ discretely fibered over $\mathbf{C}$; this stability would *not* hold if for example we had required the monomorphism to be split (i.e. admit retractions). Such stability smoothes our work, since anything proved for all $\mathbf{C}$ in a stable class can then be applied to all $\mathbf{C}/B$ for some $\mathbf{C}$ of particular interest. Since here we are interested in proving something about $\widehat{\mathbf{C}/B}$, this means that we can concentrate on studying the reduction process without at the same time combining that with a study of the particular "comma category" construction.

The stable class on which I will concentrate is the restricted one of "nodal" categories. By this I will mean any category $\mathbf{C}$ in which there exists some class $\mathcal{P}$ of objects such that

1. $C \underset{q}{\overset{p}{\rightrightarrows}} P \;\&\; P \in \mathcal{P} \Rightarrow p = q$;
2. Every map with codomain in $\mathcal{P}$ has a section;
3. Every map which does not have a section factors through $\mathcal{P}$.

It is then clear that every map in a nodal category can be factored as a split epimorphism followed by a monomorphism, since indeed *any* map with domain in $\mathcal{P}$ is (almost vacuously) a monomorphism. But this has been achieved in a very extreme way, since in fact

PROPOSITION. *In a category* $\mathbf{C}$ *nodal with respect to* $\mathcal{P}$, *every map is either an isomorphism or a* $\mathcal{P}$*-constant (in the sense of belonging to the bi-ideal of maps which factor through* $\mathcal{P}$*).*

For if p is not $\mathcal{P}$-constant, it has a section s; if s itself has a section, then that section is p and p is an isomorphism; alternatively s is $\mathcal{P}$-constant and so $s = iq$ where q has a section t because the codomain of q is in $\mathcal{P}$, but on the other hand q is a monomorphism since s is, so $tq = 1$. This shows that in the latter alternative the codomain of p is isomorphic to a $\mathcal{P}$-object, hence p itself is $\mathcal{P}$-constant.

Thus for a nodal category there is a third description of $\mathbf{C} \longrightarrow \widehat{\mathbf{C}}$: invert all maps which are both $\mathcal{P}$-constant and split epimorphisms. This smaller class of denominators may be thought of as "$\mathcal{P}$-degeneracies". Note that all $\mathcal{P}$-constant maps f really are constant in the sense that $fx_1 = fx_2$ whenever defined. Note also that if $C \longrightarrow P_1$ and $C \longrightarrow P_2$ are two degeneracies of the same C, then $P_1 \approx P_2$, that is that to a "degenerate object" there corresponds an essentially unique "$\mathcal{P}$-point".

For example the idempotent closure of the monoid of affine linear functions $ax + b : \mathbf{R} \longrightarrow \mathbf{R}$ is a nodal category.

A much smaller class which includes most of our examples are those nodal categories in which every *non-identity* map is $\mathcal{P}$-constant for a suitable $\mathcal{P}$. This remark should reinforce the reader's suspicion that much of the explicit part of this article may deserve the title "the theory of constant maps". However, the *actions* of constant maps apparently are not quite so trivial as might appear at first, somewhat as in continuum mechanics the "freezes" in configuration space often accompany serious activity in the state space. The category $\overline{\Delta}_1 = \overline{\mathbf{M}(2)}$, involved in reflexive directed graphs belongs to this subclass of the nodal categories, but $\overline{\mathbf{F}}$, where $\mathbf{F}$ is the monoid of endomaps of a two-element set, is nodal though it has non-identity isomorphisms.

PROPOSITION. *If* $\mathbf{C}$ *is nodal and* $B \in \mathcal{S}^{\mathbf{C}^{op}}$, *then* $\mathbf{C}/B$ *is also nodal.*

PROOF. Let $\mathcal{P}_B$ be the class of all objects of $\mathbf{C}/B$ whose image under the forgetful "domain" functor $\mathbf{C}/B \longrightarrow \mathbf{C}$ is in $\mathcal{P}$, where $\mathcal{P}$ is chosen so that $\mathbf{C}$ is $\mathcal{P}$-nodal. Then $\mathcal{P}_B$ surely has the property 1) of nodality. (Note that many more objects of $\mathbf{C}/B$, namely those which are non-singular figures of B, will also have property 1); the letter $\mathcal{P}$ was chosen because in case $\mathbf{C} = \overline{\mathbf{M}(T)}$ with $\mathcal{P} = (1)$, $\mathcal{P}_B$ is the subcategory of *points* of B). As for property 2), it follows from the much more general fact that for any $\mathbf{C}$, if a map $\mathbf{C}/B$ lies over a split epimorphism in $\mathbf{C}$, then it is split in $\mathbf{C}/B$ by the "same" map. Property 3) is equally easy to lift: a map in $\mathbf{C}/B$ which is not an isomorphism factors in $\mathbf{C}$ through $\mathcal{P}$, but this immediately constructs an object in $\mathcal{P}_B$ through which both portions of the factorization continue to live in $\mathbf{C}/B$.

Thus to achieve our present goal it suffices to prove

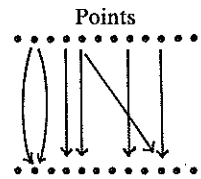
THEOREM. *If* $\mathbf{C}$ *is a nodal category then every morphism in* $\widehat{\mathbf{C}}$ *is a monomorphism.*

But in fact that is true for a vacuous reason: Every non-isomorphism in $\widehat{\mathbf{C}}$ has domain P in $\mathcal{P}$, and in fact also has codomain *not* in $\mathcal{P}$, since any map

between objects of P is already an isomorphism in $\mathbf{C}$. But any two maps $C \rightrightarrows P$ were already equal in $\mathbf{C}$, so certainly in $\widehat{\mathbf{C}}$, so any map with domain P is a monomorphism, even in $\widehat{\mathbf{C}}$.

Now in applying the above to our two object example $\overline{\mathbf{M}(T)}$, note that the objects of the category $\overline{\mathbf{M}(T)}/B$ are of two kinds, points P_B and "stars", and that for any star there are T maps to it from points expressing the vertex relations; but there are also, for the *degenerate* stars maps with label I degenerating them to actual vertices, with the resulting T idempotents from each degenerate star to itself, and it is these latter which collapse to identities in $(\overline{\mathbf{M}(T)}/B)^\wedge$.

Since the degenerate stars have been now identified with their associated points and hence may be omitted, up to equivalence the latter category may thus be pictured in two levels with all maps going down and in particular no non-identity maps to any *point*



This category will itself be a poset if every star is non-singular, and in any case is "locally" a poset as required.

Now for given T, let $B_{\mathbf{U}}$ be the object I of $\mathcal{S}^{\mathbf{M}(T)^{op}}$. Then there is just one star, and it is non-singular, so

$$(\overline{\mathbf{M}(T)}/B_{\mathbf{U}})^\wedge = \mathbf{U}(T)$$

On the other hand, let $B_{\mathbf{P}}$ be the right $\mathbf{M}(T)$-set which has just one star and also just one point, so that every $t \in T$ must act the same; in other words the star is completely singular but not degenerate (so $B_{\mathbf{P}}$ = the generic loop in the case $T = 2$ of reflexive graphs). Then

$$(\overline{\mathbf{M}(T)}/B_{\mathbf{P}})^\wedge = \mathbf{P}(T)$$

the "parallel process". In the case $T = 2$, this means that irreflexive graphs may be identified with reflexive graphs equipped with a map with discrete fiber to the loop.

Thus we have shown that for each T, each of the petit toposes $\mathcal{S}^{\mathbf{U}(T)^{op}}$, $\mathcal{S}^{\mathbf{P}(T)^{op}}$ are but two examples, corresponding to two particular objects B of the gros topos $\mathcal{S}^{\mathbf{M}(T)^{op}}$, of a family of petit toposes associated to all the objects B. Each such petit topos is actually equivalent to a subcategory

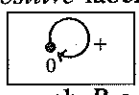
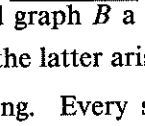
$$\mathcal{S}^{\mathbf{M}(T)^{op}}/B \rightleftarrows \mathcal{S}(B)$$

which is both reflective and coreflective, with the reflection being just the left Kan extension of $\overline{\mathbf{M}(T)}/B \longrightarrow (\overline{\mathbf{M}(T)}/B)^\wedge$. Which subcategory is it? In general if $\mathbf{C} \longrightarrow \mathbf{D}$ is surjective or a fraction construction, then $\mathcal{S}^{\mathbf{C}^{op}} \hookleftarrow \mathcal{S}^{\mathbf{D}^{op}}$ is

the subcategory in which **D**-identified actors in **C** act the same or **D**-inverted actors in **C** act bijectively. Since in our example $\mathcal{P} = \{1\}$ and $\mathcal{P}_B$ is the inverse image in $\overline{\mathbf{M}(T)}/B$, the condition that $X \longrightarrow B$, an object of $\mathcal{S}^{\mathbf{M}(T)^{op}}/B$ belong to $\mathcal{S}(B)$ is just that it be "orthogonal" to $\mathbf{I} \longrightarrow 1$.

$$\begin{tikzcd} \mathbf{I} \arrow[r] \arrow[d] & X \arrow[d] \\ 1 \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

in other words, that the fibers be discrete.

We could invert still more maps, for example those labeled by a fixed subset of T (as we did in the case $T = 2$ to obtain the free categories $\mathcal{F}$) to obtain a finer notion of the petit topos associated to the objects B. Further, there is sometimes a Grothendieck topology functorially associated to the objects B, giving a system of petit *sheaf* toposes. For example if our gros determiner is $\mathbf{C} = \mathbf{M}(2)/C$ where C is the loop, and hence the B's are the simplest kind of *labeled* graph, we can define a notion of covering in $\mathcal{F}(B)$ by declaring that all maps of *positive* label are coverings. (Here we picture the basic label values as $C =$  .) The resulting notion of petit topos assigns to each C-labeled graph B a generalized Jónsson-Tarski topos $\mathrm{sh}(B)$, the original example of the latter arising from $B =$  , two loops with the non-degenerate labeling. Every such $\mathrm{sh}(B)$ is an étendue locally homeomorphic to a generalized Cantor space $\mathcal{X}(B)$.

Finally, why should one be interested in petit toposes when we may as well study $X \longrightarrow B$ without the above restriction, and gros toposes seem to do better at englobing the study of geometrical objects? One reason is that we often want to consider actions as processes. Once a process f is *specified*, its parts must have been specified too. Thus if $X_0 \xrightarrow{f} X_1$ can be factored into stages

$$X_0 \xrightarrow{f_0} \bullet \underset{w}{\overset{u}{\rightrightarrows}} \bullet \xrightarrow{f_1} X_1$$

so that $uf_0 = wf_0$ and $f_1u = f_1w$, then $u = w$; this seems to be inherent in the notion of a *given* process. It will be guaranteed if all maps in the category of states and processes are monomorphisms, as Grassmann 1844 §8 pp. 40–41 seems to imply. On the other hand, the (self-dual) more general condition just stated *also* defines an epireflective subcategory of Cat; a topos defined by a site satisfying it is locally definable by a poset site which lives (not in $\mathcal{S}$ but) in the atomic Boolean topos $\mathcal{S}'$ of "combinatorial functors" on the category of finite sets and monomorphisms studied by Myhill and Schanuel. Because of the latter result (proved by Johnstone in his work on "QD" toposes), the Myhill-Schanuel topos $\mathcal{S}'$ seems destined to play a role in such attempts to further broaden the concept of "petit" (i.e. to generalize the

notion of space while retaining some space-like features): while internally the typical objects of S' are natural combinatorial constructions such as binomial coefficients, externally S' can be characterized by the model-theoretic job it does among all S-toposes, which is to classify infinite decidable objects; many of its remarkable properties in both these roles can be deduced from Schanuel's discovery that it is atomic in the sense of Barr, which means that $S' \longrightarrow S$ is a generalized local homeomorphism in the sense that the inverse image preserves all higher order logic. S' is petit in our generalized sense for the opposite reason from all our examples: it is defined by a site $\mathbf{C}$ (= the opposite of the category of finite sets and monomorphisms) in which all maps are *epimorphisms* in $\mathbf{C}$.

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