A Study of Function

Recently I came across a paper, according to which retrieving the material that's been read once is equivalent to re-reading the material 5 times or so. It seemed to make some sense in the sense retrieval inevitably highlights what I forgot, which in turn forces me to formulate questions such as 'what's that condition that has to be satisfied in order for the composite of two functions to be defined?', which in turn focuses my attention and help structure and glue the material to be learned into a coherent and cohesive unity.

Be that as it may, I thought of retrieving what I have been discussing for the past few weeks. Here I go. We have been talking a lot about functions such as

$$f: A \rightarrow B$$

also depicted as

$$A \xrightarrow{f} B$$

where A is the domain set and B is the codomain set. Even though 'A' and 'B' are depicted as disconnected from the arrow representing function, they i.e. domain A and codomain B are integral to the function f just as the end-points of a line-segment are integral to the line-segment. Yet another useful metaphor to keep in mind when thinking about functions is to think of a function as a journey 'j' with domain and codomain of the function corresponding to beginning (e.g. La Jolla) and destination (e.g. Amsterdam) of the journey,

$$j: \text{La Jolla} \rightarrow \text{Amsterdam}$$

also depicted as

$$\text{La Jolla} \xrightarrow{j} \text{Amsterdam}$$

We also noted that domain and codomain sets, and sets in general can be identified with identity functions such as $1_A: A \rightarrow A$, which when translated to our line-segment metaphor says that the end-points of a line-segment can be thought of as line-segments of zero length. In terms of our journey metaphor, the beginning and destination can be thought of as journeys that go nowhere (or stay wherever they are; $1_{\text{La Jolla}}: \text{La Jolla} \rightarrow \text{La Jolla}$). When we put down or formalized our thought of thinking of a set as an identity function, we found ourselves on the one hand simplifying the conceptual repertoire needed to speak of functions; on being able to speak of functions in terms of functions alone, albeit special functions, in the sense we can now say that a function has an identify function as domain and an identity function as codomain as in,

$$f: 1_A \rightarrow 1_B$$

and on the other hand confronting a problem as depicted below:

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

Looking at the above diagram, in an effort to make sense of it, one immediate question we had was 'how do we put-together or compose two functions?' Here again we found that our journey metaphor is instructive. To elaborate, consider two journeys

j: La Jolla → Amsterdam

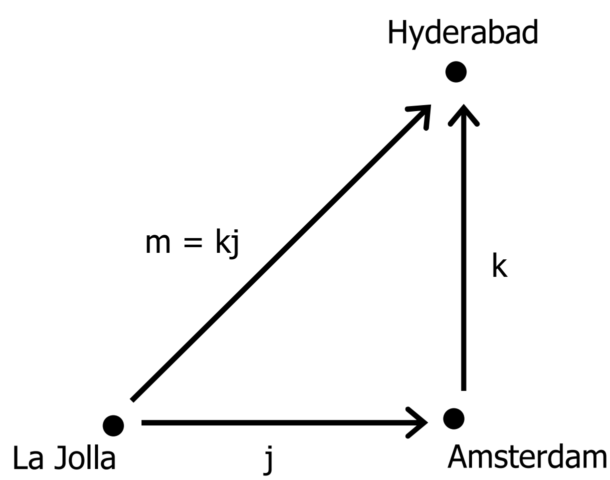
and

k: Amsterdam → Hyderabad

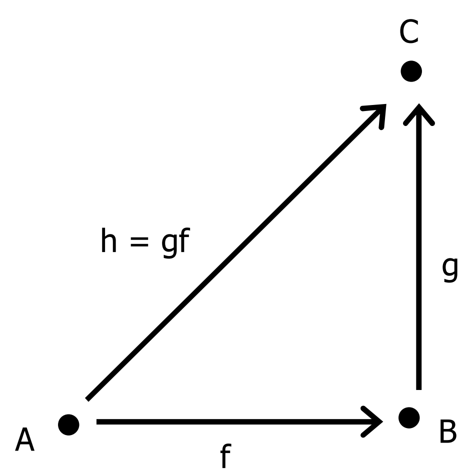
the composite kj (read as journey k after journey j) is, taking the most obvious take on journeys, the journey from La Jolla to Hyderabad. We also noted that the composite journey

kj: La Jolla → Hyderabad

of two journeys such as j and k is possible if and only if the destination of the first journey j, Amsterdam, is the same as the beginning of the second journey k, Amsterdam. Pictorially we can depict as follows:



Finally we noted that taking the journey j from La Jolla to Amsterdam and journey k from Amsterdam to Hyderabad is same as taking the composite journey m from La Jolla to Hyderabad. Now let's translate these everyday intuitions into the terminology of functions. Drawing on the above metaphor, we say that the composite of two functions f: A → B and g: C → D is defined if and only if the codomain set of the first function is same as the domain of the second function i.e. B = C, and that the domain of the composite is same as the domain of the first function and the codomain of the composite is same as the codomain of the second function. More explicitly the composite of f: A → B and g: B → C is gf: A → C.



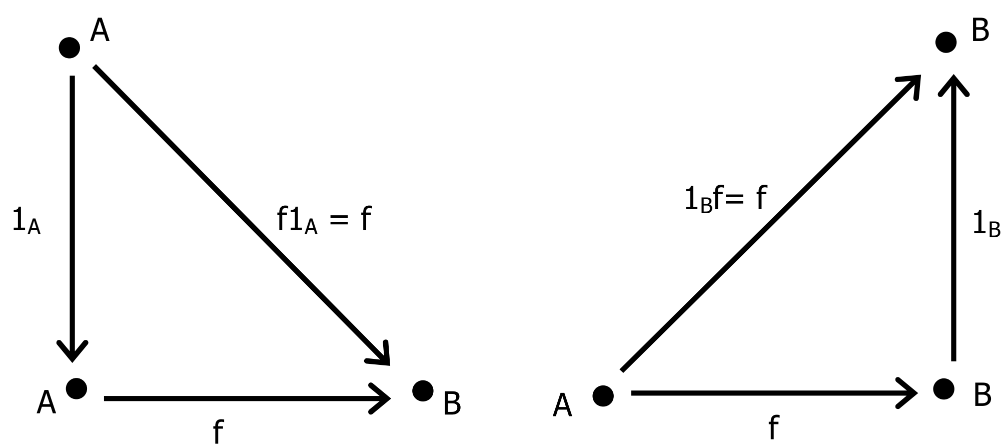
Most importantly, the composite function h is equal to the 'function g after function f'. Now we find ourselves ready to answer the question raised by our representation of function f: A → B

$$A \xrightarrow{f} B$$

as

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

under the pretext of terminological austerity. We calculated the composites $f1_A$: A → B and 1_Bf: A → B and found that $f1_A = f$ and $1_Bf = f$ as depicted below:



Now given that $f1_A$: A → B and 1_Bf: A → B are defined, we found that the composite 1_Bf1_A: A → B can be defined. Given that the following two pair-wise composites

$$A \xrightarrow{1_A} A \xrightarrow{f} B \qquad A \xrightarrow{f} B \xrightarrow{1_B} B$$

are defined, it is easy to see that the composite of all three functions is defined by way of imagining f: A → B segments of the above two pair-wise composites overlap (which is somewhat analogous to the condition that codomain of the first function f must coincide with the domain of the second function g in order for the composite gf to be defined) so that we get

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

We can also be more specific and state that the composite 1_Bf1_A can be evaluated either by first evaluating $f1_A$, which is f which when composed with 1_B gives f as the composite, which is exactly what we get when we first evaluate 1_Bf and then compose the composite f with 1_A. Or even more explicitly the composite 1_Bf1_A can be calculated either as a composite of

$f1_A: A \to B$ and $1_B: B \to B$

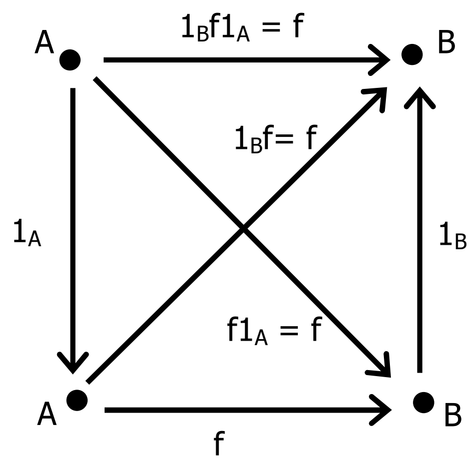
or as a composite of

$1_A: A \to A$ and $1_Bf: A \to B$

and both ways of calculating 1_Bf1_A give the same result i.e.

$$1_B(f1_A) = (1_Bf)1_A = 1_Bf1_A = f$$

as shown below:



Generalizing from identity functions to functions in general, we note that whenever two composites gf and hg are defined, then the composite hgf is defined, which can be thought of as a generalization of given 'B = C' the composite gf: A → C of functions f: A → B and g: C → D is defined, and can be calculated as the composite of gf and h i.e. h(gf) or as the composite of f and hg i.e. (hg)f is as illustrated below in terms of our favorite journeys.

j: La Jolla → Amsterdam

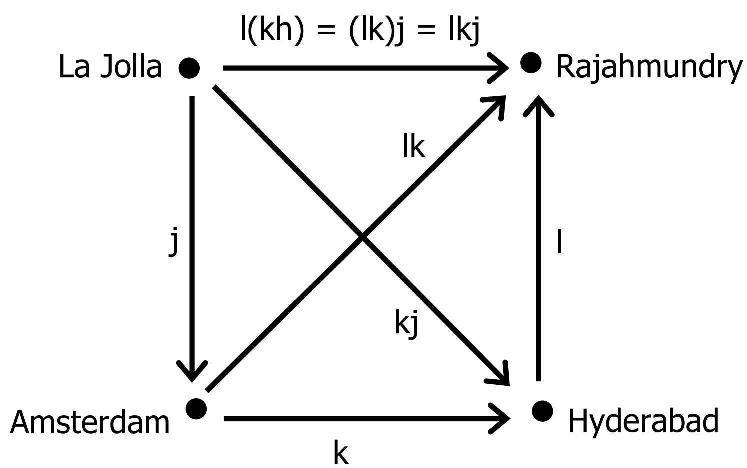
and

k: Amsterdam → Hyderabad

and

l: Hyderabad → Rajahmundry

Since the journey l's beginning is Hyderabad, which is the same as the destination of journey k, whose beginning is Amsterdam, which is the same as the destination of journey j, we can clearly form pair-wise composites (lk)j and l(kj) to obtain lkj, with, of course, (lk)j = l(kj) = lkj, when the l, k, and j are interpreted as functions.



Now let's collate our recollections of the properties of function—properties that are true of all functions—each and every function.

1. Function $f: A \rightarrow B$ has a domain A and codomain B, which are identity functions $1_A: A \rightarrow A$ and $1_B: B \rightarrow B$, respectively
2. Given two functions $f: A \rightarrow B$ and $g: C \rightarrow D$ composition of f and g is defined if B = C and the composite $h: A \rightarrow C$ is given by h = gf
3. Composite of a function $f: A \rightarrow B$ with identities $1_A: A \rightarrow A$ and $1_B: B \rightarrow B$ satisfies: $f1_A = f = 1_Bf$

4. Given three functions f: A → B, g: C → D, and h: E → F, the triple composite hgf: A → F is defined if the pair-wise composites gf: A → D and hg: C → F are defined, or in other words hgf is defined if B = C and D = E and is given as h(gf) = (hg)f = hgf

Now let's give a name to the collection of the above list of properties; since they are dealing with functions and only functions, let's call the structure formed of this list a category of functions (in naming 'category of functions' instead of 'category of sets', I seem to think I am following Ehresmann's naming convention, which is more revealing of the category we are dealing with). If we replace function with arrow to denote anything that satisfies the above 4 conditions, we have an arbitrary category or a generic category. With little rewriting we have the general notion of CATEGORY defined:

1. Arrow f: A → B has a domain A and a codomain B, which are identity arrows 1_A: A → A and 1_B: B → B, respectively
2. Given two arrows f: A → B and g: C → D composition of f and g is defined if B = C and the composite h: A → C is given by h = gf
3. Composite of an arrow f: A → B with identities 1_A: A → A and 1_B: B → B satisfies: $f1_A = f = 1_Bf$
4. Given three arrows f: A → B, g: C → D, and h: E → F, the triple composite hgf: A → F is defined if the pair-wise composites gf: A → D and hg: C → F are defined, or in other words hgf is defined if B = C and D = E and is given as h(gf) = (hg)f = hgf

In passing we may note that, with isomorphisms, a subset of arbitrary functions, as arrows we obtain the notion of groupoid, and with automorphisms, a subset of isomorphisms, as arrows we obtain the notion of group.

It might be helpful to state what we mean by a CATEGORY in plain English. A CATEGORY, in plain English, is a mathematical universe or a domain of mathematical discourse. For example, the category of functions that we were talking about in this session is the mathematical universe inhabited by sets, functions, and composition of functions. Alternatively, the category of functions is a mathematical discourse about sets, functions, and composite of functions. In a sense the mathematical notion of CATEGORY is not inconsistent with its everyday usage.

In the spirit of complete disclosure, since I am not so sure about the legitimacy of the way we arrived at the notion of CATEGORY as a collection of properties of functions, I'll go over the textbook definition of CATEGORY, which on the surface does not seem to be much different, but may differ in matters that matter most.

Before we close let's look at a concrete illustration of the notion of CATEGORY, especially one in which arrows are not functions (Arbib & Manes). Before we get to the category, we need to have a definition in place.

A poset (or partially ordered set) is a set A with a structure of ≥, which is

Reflexive: $a \geq a$ for all a in A

Antisymmetric: $a \geq a'$ and $a' \geq a \Rightarrow a = a'$ for all a, a' in A

Transitive: $a \geq a'$ and $a' \geq a'' \Rightarrow a \geq a''$ for all a, a′, a″ in A

Consider the set A = {1, 2, 3, 4} along with the structure '≥', so that we have as arrows $2 \geq 1$, $3 \geq 2$, etc., where 1, 2, 3, and 4 are considered objects or identity arrows. The identity arrows such as $4 \geq 4$ are given by the reflexivity of the structure of ≥. The composite of two composable arrows $3 \geq 2$ and $2 \geq 1$ is $3 \geq 1$ by virtue of transitivity of ≥, and is in accord with the definition of composition of arrows of a category. We can also note that the composite of an arrow with its identities is the arrow as in the composite of $3 \geq 3$ and $3 \geq 2$ is $3 \geq 2$, and the composite of $3 \geq 2$ and $2 \geq 2$ is $3 \geq 2$. Having checked the identity laws, let's check to see if associativity holds true. The composite of three composable arrows: $4 \geq 3$, $3 \geq 2$, and $2 \geq 1$ can be evaluated by first evaluating the composite of $3 \geq 2$, and $2 \geq 1$, which is $3 \geq 1$, and then evaluating the composite of $4 \geq 3$ and $3 \geq 1$, which is $4 \geq 1$. Alternatively, we could first evaluate the composite of $4 \geq 3$ and $3 \geq 2$, which is $4 \geq 2$, and then evaluate the composite of $4 \geq 2$ and $2 \geq 1$, which is $4 \geq 1$; thereby upholding associativity. Thus we have a category (a poset) in which $a \geq b$ is an arrow (and not a function) and $a \geq a$ is the identity arrow on a in A. This example clearly shows that arrows of category need domain and codomain, which could be identities, and as long as there is composition of arrows defined satisfying identity and associative laws, we have a category.