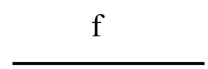
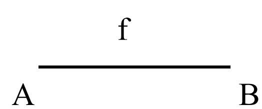
A Line and Its Two Endpoints

Let's look at a line

Let's denote the line-segment with f as in:



Looking at the line-segment f we notice that f has 2 endpoints: one on the left and one on the right. Let's use A to denote the endpoint on the left and B to denote the endpoint on the right as shown below:



Given that our objective is clarity—clarity of thought—to be contrasted with clutter—cluttering our mind with words and verbiage, we are naturally inclined to reflect, however small an iota we can afford, on the relation between line segment f and endpoints A and B, i.e., line and point. Line has length—a property with which we are quite familiar. We can readily think of our line-segment f as having some length, say f cm. How about endpoints A and B? What do they have? Let's entertain the thought that point is a line-segment of 0 length (e.g. 0 cm [1]). Now we have a line-segment f (of some length) with two line-segments A and B of 0 length as endpoints. Notwithstanding the fact that treating point as a line of 0 length accords with our visual sensibilities, what are we going to do with the fact that the line f has two endpoints A and B. If points A, B are lines (albeit of 0 length), then ‘what are the endpoints of points (or lines) A, B?’ Now it might help to get physical. Let’s think of the line segment f as a path from point A to point B or as a journey of distance f km from a town A to a town B as depicted below

$$A \xrightarrow{f} B$$

If f is a journey that takes us from A to B with the distance from A to B as f km, then we imagine the origin of the journey f—the point A—as a journey that goes from A to A, with the distance from A to A as 0 km. Similarly, we think of the destination of the journey f—the point B—as a journey that goes from B to B, with the distance from B to B, again, as 0 km. Pictorially,

$$A \longrightarrow A \xrightarrow{f} B \longrightarrow B$$

Denoting the path that goes from A to A as 1_A (for now let’s treat this just as some convention) and the path that goes from B to B as 1_B, we have

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

Let's look at what we got here:

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

We have 3 lines: 1_A, f, and 1_B and each one of the lines has 2 endpoints (beginning and end), so we have a total of 6 endpoints for 3 lines. But, looking above we see 4 endpoints for the 3 lines; aren't we missing 2 endpoints [2]?

OK, looking at the big picture, to begin with we started with a line

$$A \xrightarrow{f} B$$

and ended up with 3 lines

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

Thanks to all the '='s we are used to seeing in equations—in math classes, we are, sooner or later, likely to wonder if

$$A \xrightarrow{f} B$$

is equal to

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

and making an equation of the above scenario, we ask:

If

$f = 1_A f 1_B$

is true?

To begin with we are not quite sure what to make out of the equation

$f = 1_A f 1_B$

but we are familiar with somewhat similar situations such as

$3 = 0 + 3 + 0$

or

$5 = 1 \text{ x } 5 \text{ x } 1$

which doesn't necessarily tell us if

$f = 1_A f 1_B$

is true, but at least our question

'Is

$f = 1_A f 1_B$

true?'

may not be totally loony and may very well be worth asking. So let's ask:

Is '$f = 1_A f 1_B$' true?

Or pictorially,

Is

$$A \xrightarrow{f} B$$

equal to

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

In an attempt to answer the question

Is '$f = 1_A f 1_B$' true?

we looked at equations such as

$x = 0 + x + 0$

and

$y = 1 * y * 1$

and saw the sensibility of our own equation

$f = 1_A f 1_B$

without, of course, assigning true or false to our equation.

Equations such as

$x = 0 + x + 0$

and

$y = 1 * y * 1$

didn't answer our question:

Is '$f = 1_A f 1_B$' true?

but they do suggest a question answering which might, in turn, help us answer our original question:

Is '$f = 1_A f 1_B$' true?

When we look at situations such as

$x = 0 + x + 0$

we recognize that the sum of 3 numbers (0, x, and 0) is a number (x), and when we look at scenarios such as

$y = 1 * y * 1$

we see that the product of 3 numbers (1, y, and 1) is a number (y), which suggests the following question:

What is it that we are looking at when we are looking at the concatenation:

$1_A f 1_B$

of 3 lines

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

Is

$1_A f 1_B$

supposed to be the sum of 3 lines (1_A, f, and 1_B)

or the product of 3 lines

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

Well, for starters, heeding the advice of Sontag (c.f. Against Interpretation), let's take it for what it is or treat

$1_A f 1_B$

as what

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

is, i.e., $1_A f 1_B$ is the concatenation or putting together of 3 lines 1_A, f, and 1_B.

Now we get to ask, 'can the concatenation of 3 lines (1_A, f, and 1_B) equal a line (f)?'

Let's now go back to our elementary school addition

$0 + 3 + 0 = 3$

Given

$0 + 3 + 0$

how do we get to

3

Given

$0 + 3 + 0$

we first, starting from left, evaluate pair-wise

$0 + 3 = 3$

to get

$3 + 0$

which we evaluate to finally get

3

Alternatively, given

$0 + 3 + 0$

we can begin from right and evaluate, again, pair-wise

$3 + 0 = 3$

to get

$0 + 3$

which we evaluate to finally get, as earlier,

3

In both cases (with different orders of evaluation: one from left and the other from right), we evaluate (eventually) the sum of 3 numbers by evaluating the sum of 2 numbers at a time.

The moral of this number story for our line story, for now; more later, is that in order to answer the question of how do we evaluate the concatenation

$1_A f 1_B$

of three lines (1_A, f, and 1_B), we first need to address the question of ‘how do we go about concatenating or putting together 2 line-segments into a line-segment?’

Here again our everyday, run-of-the-mill, mundane, commonplace, experience is quite enlightening (as is often the case if only we paid it attention instead of the Chopras and Swamis who loot us—our senses—silly, nilly).

Let’s say I embark upon a journey, with my itinerary involving 2 buses, going from point A to point C with the first bus f taking me from point A to point B and the second bus g taking me from point B to point C as shown below

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Alternatively, we can depict the totality of my journey as

$$A \xrightarrow{fg} C$$

In the above we began with two paths or line-segments

$$A \xrightarrow{f} B$$

and

$$B \xrightarrow{g} C$$

which we put-together (in the course of our journey) into the line-segment

$$A \xrightarrow{fg} C$$

Here we clearly did put-together 2 line-segments into a line-segment, but how? Let's say I want to go from San Diego to Bangalore. Since there are no direct flights from San Diego to Bangalore, I take two flights. I have limitless options (which is what drives me crazy), but there is one thing I have to adhere to religiously, if I were to ever get to Bangalore from San Diego, in picking my two flights. I can go from San Diego to Los Angeles, or Atlanta, or Chicago…, but to get to Bangalore my second flight must be from wherever my first flight landed. Even more explicitly, if I go from SD to LA, then my next flight must leave from LA, which is exactly the condition for concatenating 2 line-segments:

$$A \xrightarrow{f} B$$

and

$$B \xrightarrow{g} C$$

into

$$A \xrightarrow{fg} C$$

In other words, we can concatenate 2 line-segments

$$A \xrightarrow{f} B \qquad C \xrightarrow{g} D$$

into the line-segment

$$A \xrightarrow{fg} D$$

if end-point of f, B = starting-point of g, C.

Now if I take a flight f from A to B and another flight 1_B from B to B, then I go from A to B, which is the same as f. Pictorially,

$$A \xrightarrow{f} B \xrightarrow{1_B} B$$

is equal to

$$A \xrightarrow{f1_B} B$$

which, in turn, is equal to

$$A \xrightarrow{f} B$$

Continuing with my flights of fancy, if I take a flight 1_A from A to A and then take a flight from A to B, then, again, I go from A to B, which is, again, the same as f.

$$A \xrightarrow{1_A} A \xrightarrow{f} B$$

is equal to

$$A \xrightarrow{1_A f} B$$

which, in turn, is equal to

$$A \xrightarrow{f} B$$

By now I wouldn't be surprised if we lost track of how and why we got into all of this.

We began with a line-segment

$$A \xrightarrow{f} B$$

and ended up with

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

which led us to ask

Is

$$A \xrightarrow{f} B$$

equal to

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B$$

In terms of equations,

Is '$f = 1_A f 1_B$' true?

In the above equation we have a line-segment f and a concatenation of three line-segments $1_A f 1_B$. We can evaluate the concatenation of three lines

$$1_A f 1_B$$

two at a time, either, from left to right, by first evaluating

$$1_A f = f$$

and then evaluating

$$f 1_B = f$$

Alternatively, we can evaluate the concatenation

$$1_A f 1_B$$

pair-wise, from right to left, by first evaluating

$$f 1_B = f$$

and then evaluating

$1_A f = f$

Either way, we find that

$1_A f 1_B = f$

Now if we formalize the above everyday experience with a line and its two endpoints, then we get to define CATEGORY (see page 21 of Conceptual Mathematics textbook), which we'll do in a subsequent note.

[1.]. We Indians love zero; we think we invented 0, and me, along with we, took great pride in it until I got smart and realized that mankind had the concept of NOTHING long before ZERO, therefore we Indians need not get all that worked up about inventing 0 only to be humbled numb when I tried to think about what does it take, conceptually speaking, to end up with 0 given nothing.

[2]. I am tempted to go off on a tangent and talk about Schanuel's 'length of a potato,' but let's practice discipline and stick to the task at hand.